Elementary Theory of Del Pezzo Surfaces

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Abstract. Del Pezzo surfaces are certain algebraic surfaces in projective n-space of degree n. They contain an interesting configuration of lines and have a rational parametrization. We give an overview of the classification with an emphasis on algorithmic constructions (e.g. of the parametrization), on explicit computations, and on real algebraic geometry.

1 Introduction

This paper is elementary in the sense that it does not use the concepts and terminology of modern algebraic geometry, such as sheaves, schemes, divisors, or vector bundles. My personal opinion is that these concepts belong more to the "algebraic" than to the "geometric" part of "algebraic geometry", and the goal was to write an introduction to Del Pezzo surfaces for geometers and not for algebraists. This is also the reason why the paper is of survey type, but it cannot be used as an introduction to the modern theory of Del Pezzo surfaces. From that point of view, the main interest in Del Pezzo surfaces is related to birational classification of algebraic varieties of higher dimension (e.g. Calabi-Yau threefolds) or to arithmetic questions, and these relations are not even touched upon here. Our main intention was to collect material about this classical topic which could be of some interest to applied geometers. The main emphasis has been put on algorithmic techniques and on examples. For this reason, it would have been more justified to give the title "a very biased look at Del Pezzo surfaces".

The paper does assume a good familiarity with projective geometry, and the described algorithmic techniques can only be carried out if one can solve systems of algebraic equations in several unknowns.

The definition of Del Pezzo surfaces given in sect. 4 is not the usual one (which uses canonical divisors), but it follows Del Pezzo [5], who encounters this class of surfaces in his investigation of surfaces of degree n in $\mathbb{P}^n$. In the course of arriving at this definition, we give some theorems (Theorem 2, Theorem 5, Theorem 8, and Theorem 11) and occasionally proofs. Of course, these theorems are classical facts whose origins date back by centuries. A proof of Theorem 5 can be found in [7].

The unprojection algorithm in sect. 5 is original. Its advantage is that it makes a uniform treatment of parametrization algorithms (see sect. 6) possible.

The classification of Del Pezzo surfaces in sect. 6 is due to [5]; a modern proof can be found in [12]. No proof is contained in this paper because it would be too long and too technical. A complete elementary proof of Theorem 17 would also be surprisingly complicated because the finiteness of resolution of singularities is not a priori clear. Of course, the theorem also follows from the classification given in [5].

In the chosen approach, Del Pezzo surfaces of degree 2 and 1 are certainly unnatural (they also do not arise in [5]). But as early as in [10], these cases are discussed together with the other Del Pezzo surfaces, in the context of the classification of linear systems of elliptic curves in the plane. Theorem 24 and Theorem 28 can be found in [4].

The real classification of Del Pezzo surfaces, especially Theorem 30, is due to [3]. Modern treatments can be found in [20, 15, 21]. The technique used in example 35 to compute an improper parametrization (see also remark 36) is also mentioned in [3, 13, 17].

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2 Projective Varieties, Degree, and Projection

Let $\mathbb{P}^n$ denote complex projective space of dimension n. Let $X \subset \mathbb{P}^n$ be a projective algebraic variety, i.e. the zero set of a homogeneous prime ideal. The *dimension* of X can be defined as the smallest integer m such that there exists an $n-m-1$-dimensional linear subspace disjoint from X. A generic linear subspace of dimension $n-m$ intersects X in a finite number of points. If we count with multiplicities, then this number depends only on X, and this is a way to define the *degree* of X (following [8]).

Let $p \in \mathbb{P}^n$ be a point, e.g. $p = (x_0:\ldots:x_n) = (1:0:\ldots:0)$ (the affine origin). Let H be a linear hyperplane not containing p, e.g. the plane $x_0 = 0$. The projection $\pi_{p,H}$ with center p onto H is defined for all points except p. In the example, this is just the omission of the first projective coordinate x_0. - Let Y be the closure of the image of X under this projection. It is again a projective variety. Its dimension is either m or $m-1$. The second is the case if and only if X is a cone and p is its vertex, or X is a linear space and p is a point on X.

Remark 1. The choice of H is not essential. A different choice leads to another projective image Y' which is projectively equivalent to Y. In the following, we will often omit any explicit references to H.

If $\dim(Y) = \dim(X) = m$, then there is a positive integer f such that the preimage of a generic point of the projection map $\pi_p : X \to Y$ has f points. In case $f = 1$, then $\pi_p : X \to Y$ is birational. The number f is called the *tracing index* of the projection.

A generic linear $n-m-1$-subspace L of H intersects Y in $\deg(Y)$ points. The linear span of L and p intersects X in $f \cdot \deg(Y)$ points plus an intersection at p, that has to be counted with multiplicity

$$r := \deg(X) - f \cdot \deg(Y). \tag{1}$$

The number r is also called the *multiplicity* of X at p, and p is also called an r-fold point of X. Nonsingular points have multiplicity 1, and points outside X have multiplicity 0.

Theorem 2. *Let $X \subset \mathbb{P}^n$ be a projective variety of dimension m and degree d. Assume that X is not contained in a proper linear subspace. Then $d \geq n - m + 1$.*

Proof. We proceed by induction on n, fixing m. If $n = m$ (obviously the smallest possible value for n), then $X = \mathbb{P}^m$ and $d = 1$. The inequality is fulfilled.

Assume $n > m$. Let p be a nonsingular point of X. Let $Y \subset \mathbb{P}^{n-1}$ be the image of X under the projection from p. If Y were contained in a proper linear subspace L, then X would be contained in the linear span of L and p, contradicting the assumption. Therefore Y is not contained in a proper linear subspace.

Let f be the tracing index of the projection. Then

$$d = f \cdot \deg(Y) + 1 \geq f \cdot (n - m) + 1 \geq n - m + 1,$$

where the first inequality is a consequence of the induction hypothesis.

Remark 3. A closer look at the proof reveals that if equality holds, then the variety is rational (i.e. birationally equivalent to a projective space). Indeed, in this case the tracing index is always 1 in each projection step, so that we get a birational map from X to $\mathbb{P}^m$.

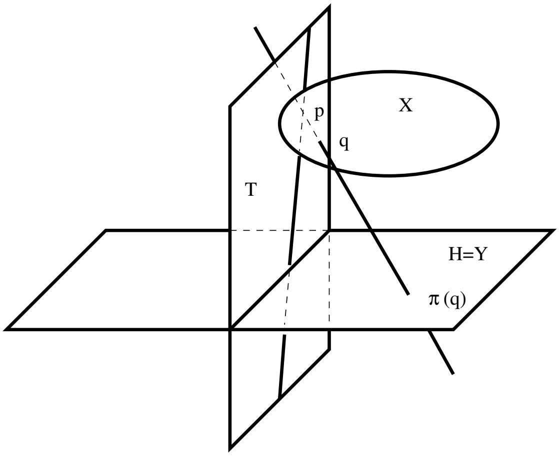

Fig. 1. Projection from a nonsingular point

Projections from nonsingular points are of special interest. Let $X \subset \mathbb{P}^n$ be a variety of dimension m, and let p be a nonsingular point of X. The projection $\pi_p : X \to Y$ is not defined at p. But for any differentiable curve $C : [0, 1] \to X$ with $C(0) = p$, the limit $\lim_{t\to 0} \pi_p(C(t))$ exists and lies on Y. The set of all these limits is equal to the intersection of the tangent space T of X at p with the projection hyperplane H (see fig. 1).

Conversely, assume that we have a variety $Y \subset \mathbb{P}^n$ of dimension m and an $m-1$-dimensional linear subspace L on it. Can we construct a variety $X \subset \mathbb{P}^{n+1}$ and a nonsingular point $p \in X$ such that Y is the image of X under the projection by p? We will give partial answers to this question below.

3 Varieties of Minimal Degree

Let $C \subset \mathbb{P}^n$ be a curve of degree n. (From now on, a statement such as $C \subset \mathbb{P}^n$ implicitly implies the assumption that C is not contained in a proper linear subspace.) By remark 3, C is rational. Therefore, C has a parametrization $(P_0(t):\ldots:P_n(t))$ with polynomials $P_0, \ldots, P_n$ of degree at most n (and the maximum is reached by at least one of the P_i).

Since C is not contained in a proper linear subspace, the P_i are linearly independant. But the vector space of all polynomials of degree at most n has dimension $n+1$, and so the P_i form a basis. We can apply a projective transformation in order to transform this basis into the standard basis $P_i = t^i$, $i = 0, \ldots, n$. This implies that, up to projective transformations, there is precisely one curve $C \subset \mathbb{P}^n$ of degree n, which is also called the *rational normal curve* of degree n.

Remark 4. The Steiner construction (see [7], p. 528–533), shows that for any $n+3$ points in general position, there is a unique rational normal curve passing through them.

For surfaces, we have a similar classification (see [7], p. 525).

Theorem 5. *Let $S \subset \mathbb{P}^n$ be a surface of degree $n-1$. Then S is either a rational scroll $R_{n,r}$ with parametrization $(1:t:\ldots:t^{n-r-1}:s:st:\ldots:st^r)$ for some $r \leq \frac{n-1}{2}$ (up to projective transformation), or $n = 5$ and S is the Veronese surface V with parametrization $(1:t:t^2:s:st:s^2)$ (up to projective transformation).*

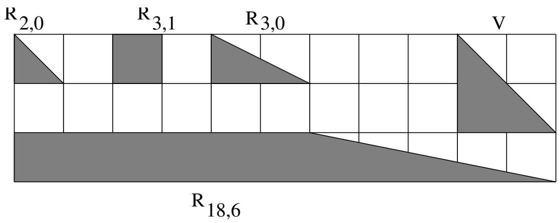

Fig. 2. Surfaces of minimal degree in lattice representation

Remark 6. Note that in both cases, the surface S is *toric*, i.e. parametrized by monomials. Toric surfaces have recently been used in [11] in order to generate multi-sided surface patches; they can be represented by lattice polygons in an obvious way. Actually, the surfaces of minimal degree are precisely the toric surfaces associated to a polygon without interior lattice points (see fig. 2).

Clearly, $R_{2,0}$ is the projective plane, and $R_{n,0}$ is the cone over the rational normal curve of degree $n-1$. The quadric surface $R_{3,1} \subset \mathbb{P}^3$ has two rulings (families of lines), namely the ruling of lines given by $s =$ constant and the ruling of lines given by $t =$ constant.

Remark 7. It is easy to see that the rational scroll $R_{n,r}$ is a projection of $R_{n+1,r}$ – just omit the coordinate corresponding to t^{n-r}. The Veronese surface is not the projection of a surface in $\mathbb{P}^6$ of minimal degree, because it does not contain any lines.

4 Curves of Almost Minimal Degree

We say that a variety $X \subset \mathbb{P}^n$ has *almost minimal degree* if $\deg(X) = n - \dim(X) + 2$.

For any $n \geq 2$, we can produce almost minimal curves $C \subset \mathbb{P}^n$ by "spoiling rational normal curves". Take a rational normal curve $C' \subset \mathbb{P}^{n+1}$ of degree $n+1$, and a point p outside C. Let C be the image of C' under the projection from p. Since p is a point of multiplicity 0, the degree of C is a divisor of $n+1$. By Theorem 2, the degree is greater than or equal to $n+1$, therefore it is equal to $n+1$.

Conversely, any rational curve of almost minimal degree $C \subset \mathbb{P}^n$ is a spoiled rational normal curve. To show this, we write down a parametrization $(P_0(t) : \cdots : P_n(t))$ be a sequence of polynomials with maximal degree $n+1$. Let $Q(t)$ be a polynomial of degree at most $n+1$ that is linearly independent of $P_0, \ldots, P_n$. Then the curve defined by $(P_0(t):\ldots:P_n(t):Q(t))$ is a rational normal curve, and C is the image of the projection from $(0:\ldots:0:1)$. (This point must be a point outside the rational normal curve because its multiplicity is zero by the degree formula 1.)

There are also irrational curves of almost minimal degree; the first examples are the nonsingular cubic plane curves. It is well-known that the nonsingular plane cubic curves are *elliptic*, i.e. of genus one. Here is a general theorem on irrational curves of almost minimal degree.

Theorem 8. *Let $C \subset \mathbb{P}^n$ be an irrational curve of almost minimal degree. Then C is elliptic and nonsingular.*

Proof. We proceed by induction. If $n = 2$, it suffices to state that cubic plane curves are either rational or elliptic, and the elliptic ones are nonsingular.

Let $n \geq 3$. Let p be a nonsingular point on C. Let C' be the image of C under projection from p. Let $d := \deg(C')$. Then $d|n$ by the degree formula 1, and $d \geq n-1$ by Theorem 2. This implies that $d = n$, i.e. C' has almost minimal degree, and the projection gives a birational map $C \to C'$. By induction hypothesis, C' is elliptic. Since the genus is a birational invariant, C is also elliptic.

In order to show that C is nonsingular, let q be an arbitrary point of C, and let r be its multiplicity. Let D be the image of C under projection from q. Then $\deg(D)$ is a divisor of $n+1-r$, which is greater than or equal to $n-1$. This leaves only the cases $r = 1$ and $\deg(D) = n$, or $r = 2$ and $\deg(D) = n-1$. In both cases, the tracing index of the projection must be one, so that C and D are birationally equivalent. But this rules out the second case, because D would then have minimal degree and therefore be rational. Hence $r = 1$, and we showed that C has only points with multiplicity one.

Example 9. For any $n \geq 2$, we have an elliptic curve $C \subset \mathbb{P}^{n+1}$ of almost minimal degree. Here is an example for $n = 3m-1$.

Let C be the plane cubic with equation $x^3 + y^3 + z^3 = 0$. Let $f : \mathbb{P}^2 \to \mathbb{P}^n$ be the embedding given by

$$(x{:}y{:}z) \mapsto (x^m{:}\ldots{:}y^m{:}x^{m-1}z{:}\ldots{:}y^{m-1}z{:}x^{m-2}z^2{:}\ldots{:}y^{m-2}z^2).$$

The image of C is of degree $3m = n + 1$.

For all other n, examples can be constructed by one or two steps of point projection of the above example.

Remark 10. In general, it is not true that the absence of singularities of a curve of almost minimal degree implies that the curve is elliptic. An example of a spoiled rational normal curve without singularities is the "twisted quartic" in $\mathbb{P}^3$ with parametrization $(1{:}t{:}t^3{:}t^4)$.

5 Surfaces of Almost Minimal Degree

We can produce surfaces of almost minimal degree by spoiling surfaces of minimal degree, as we did in the previous section for curves. These surfaces are rational, and the projections of rational scrolls are ruled surfaces.

As a base for some proofs on induction, we need to have a rough classification of the cubic surfaces in $\mathbb{P}^3$. We distinguish the following types.

1. Cubic surfaces with a double line. These are the projections of cubic rational scrolls in $\mathbb{P}^4$.
2. Cones over nonsingular cubic plane curves. These are irrational. They have a triple point and no other singularities.
   (Note that the cones over singular cubic plane curves are already falling into type 1 above.)
3. Cubic surfaces with isolated double points. These are rational. Indeed, projection from a double point gives a birational map onto $\mathbb{P}^2$.
4. Nonsingular cubic surfaces. These are also rational.

A much finer classification can be found in [2, 1].

Type 2 can easily be generalized to arbitrary dimension: the cone over an elliptic curve of almost minimal degree is an irrational surface of almost minimal degree. It is well-known [5, 6] that every irrational surface of almost minimal degree is a ruled surface with elliptic base.

We define a *Del Pezzo surface* as a rational surface of almost minimal degree that is not a spoiled surface of minimal degree. The cubic Del Pezzo surfaces are the surfaces of type (3) and (4) above.

Theorem 11. *Let S be a Del Pezzo surface.*
a) S has at most isolated double points.
b) If S has degree at least 4, then the image of S under projection from a nonsingular point $p \in S$ is a Del Pezzo surface.
c) A generic hyperplane section of S is an elliptic curve of almost minimal degree.
d) The number of lines on S is finite.

Proof. (c): It is obvious that the generic hyperplane section has almost minimal degree. They are not rational, because then the surface would be a spoiled minimal surface (this is a consequence of the discussion of surfaces with rational hyperplane sections in [4]). Hence they are elliptic.

(b): Let S' be the image of the projection. By the degree formula 1, S' has almost minimal degree and is birationally equivalent to S. The hyperplane sections are projections from intersections of S with hyperplanes through p. Because of (c), these are elliptic curves. So, S' is rational and has generic hyperplane sections of genus one. On the other hand, S' cannot be a spoiled surface of minimal degree, because these have generic hyperplane sections of genus zero. Hence S' is a Del Pezzo surface.

(a): By the degree formula 1, S cannot have points of multiplicity 3 or more. We prove that the number of double points is finite, by induction on the degree. For degree 3, this follows from the classification of cubic surfaces above. For $n > 3$, choose a nonsingular point and project; the image is again a Del Pezzo surface S' of degree $n-1$, by (b). Therefore S' has only finitely many double points, by the induction hypothesis. It follows that the number of double points on S is also finite, since the image of a double point is a double point.

(d): We proceed by induction. For degree 3, it is well-known that any cubic surface of type (3) or (4) has only finitely many lines. For $n \geq 4$, assume indirectly that S has infinitely many lines. Let p be a nonsingular point on S. Since S is not a cone with vertex p, there are only finitely many lines through p. Hence there remain infinitely many lines on the image S' of the projection from p. But S' is a Del Pezzo surface, contradicting the induction hypothesis.

The lines on a Del Pezzo surface are interesting for several reasons. One of them is that they can be used to construct a Del Pezzo surface of degree one higher which projects to the given Del Pezzo surface.

Here is an explicit *unprojection algorithm*. It assumes that we have given a Del Pezzo surface $S \subset \mathbb{P}^n$ and a line l lying on S.

1. Choose a generic linear form $L(x_0, \ldots, x_n)$ vanishing on l.
2. Compute the intersection of the hyperplane defined by L with S. As we will show in Theorem 14, it consists of two components: the line l and a rational normal curve C of degree $n-1$.
3. Choose a generic quadratic form $Q(x_0, \ldots, x_n)$ vanishing on C. (We will show in Theorem 14 that there exist such quadratic forms.)
4. Compute the image of S under the map given by

$$(x_0 : \ldots : x_n) \mapsto \left(x_0 : \ldots : x_n : \frac{Q(x_0, \ldots, x_n)}{L(x_0, \ldots, x_n)} \right).$$

Example 12. Let $S \subset \mathbb{P}^3$ be the surface given by

$$3x_0x_1^2 + 3x_0x_2^2 + 3x_0x_3^2 - 3x_0^3 - 10x_1x_2x_3 = 0.$$

This cubic has 27 lines on it (see fig. 3). Let l be the line $x_0 = x_3 = 0$.

We choose the linear form $L := x_3$. It intersects S in l and in the plane conic C defined by $x_3 = x_1^2 + x_2^2 - x_0^2 = 0$.

Now we choose the quadric $Q := x_1^2 + x_2^2 - x_0^2$.

To compute the image of the map defined in the unprojection algorithm, we introduce a new variable x_4. The equation $Lx_4 - Q = 0$ holds on the image. A second

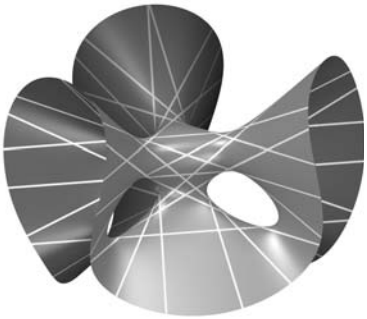

Fig. 3. A cubic Del Pezzo surface with 27 real lines (picture courtesy of O. Labs)

equation can be found by writing the equation of S as linear combination of L and Q and dividing by L, replacing Q/L by the new variable x_4:

$$\frac{3x_0(x_1^2 + x_2^2 - x_0^2) + (3x_0x_3 - 10x_1x_2)x_3}{x_3} = 3x_0x_4 + 3x_0x_3 - 10x_1x_2 = 0.$$

The image is the intersection of these two quadratic forms in $\mathbb{P}^4$, which is indeed a surface of degree 4.

Remark 13. How do we know whether our choice of the linear or quadratic form in steps 1 and 3 were general enough? In practice, the best strategy is just to try an arbitrary one. There is a chance that the choice does not work, but the bad choices are of measure zero in the set of all choices.

Theorem 14. *The unprojection algorithm is correct.*

Proof. We begin by proving the statement claimed in step 2: the hyperplane defined by L intersects S in l and a rational normal curve of degree $n-1$. In fact, it is clear that l is a component of the intersection, and that the degrees of the remaining irreducible components add up to $n-1$, but we have to show that there is only one remaining component.

Let p be a nonsingular point on l. The projection from p is a Del Pezzo surface S' by theorem 11. The line l projects to a point $q \in S'$, which is either a single or a double point (in fact, it is always a double point, as we will see in remark 15 below). Projection from q is birational by the degree formula 1: let S'' be the image. The remaining components project to a generic hyperplane section of S''. By Bertini's theorem (see [9], Thm. 8.18, p. 179; Rem. 8.18.1, p. 180), generic hyperplane sections are irreducible, and the statement is proven.

The ideal of the rational normal curve C is generated by quadratic forms. Therefore the generic quadratic form Q through C does not vanish identically on the line l. Consequently, Q is not contained in the vector space generated by the quadratic multiples of L and the quadratic forms vanishing on S. This implies that the image S_0 of the unprojection constructed in step 4 is not contained in a linear subspace.

The degree of S_0 is the number of intersections of S' with two generic hyperplanes H_1, H_2. We can assume that the form defining H_1 does not contain the new variable x_{n+1} (by linear algebra). It defines a hyperplane $H_3 \subset \mathbb{P}^n$. The intersection points of H_1 and H_2 and S_0 correspond to the intersection points of S and H_3 and some quadric surface Q_0, which we get when we multiply the equation of H_2 by L and replace L times the new variable by Q, minus the intersection points of S, L, and H_3. This number is $2n - (n-1) = n+1$. Therefore S_0 is a surface of almost minimal degree.

There is an obvious projection from S_0 to S (omitting the last coordinate). The center is a nonsingular point, by the degree formula 1. Because S_0 is rational and is not a spoiled surface of minimal degree, S_0 is a Del Pezzo surface.

Remark 15. Revisiting the above proof again, we can now show that if p is a nonsingular point lying on a line l contained in S, then the image q of l under the projection is a double point on the image S' of S. Let S'' be the image of the projection from q. The generic hyperplane section of S'' is a birational image of the rational normal curve which forms together with L the intersection of S with a general hyperplane through L. Hence S'' is not a Del Pezzo surface, and q cannot be a nonsingular point.

6 Classification of Del Pezzo Surfaces

For the theory of Del Pezzo surfaces, the techniques of projection and unprojection are very useful because they allow induction proofs (upward and downward). We can draw an (infinite) directed graph of all Del Pezzo surfaces up to projective transformations, with an edge from S_1 to its images under projections from nonsingular points. The natural question arises: is this graph connected?

It is clear that it would suffice to show that there is a path connecting any two cubic Del Pezzo surfaces, because we can always do projection steps down to degree 3, and these are the minimal vertices of the graph.

Another possible approach is to locate the maximal vertices of the graph.

Theorem 16. *Let S be a Del Pezzo surface without a line. Then S is one of the following three surfaces:*

1. *the nonsingular surface $F_9 \subset \mathbb{P}^9$ with parametrization* $(1{:}s{:}t{:}s^2{:}st{:}t^2{:}s^3{:}s^2t : st^2{:}t^3)$;
2. *the nonsingular surface $F_8 \subset \mathbb{P}^8$ with parametrization* $(1{:}s{:}s^2{:}t{:}st{:}s^2t{:}t^2{:}st^2 : s^2t^2)$;
3. *the surface $G_8 \subset \mathbb{P}^8$ with parametrization* $(1{:}s{:}s^2{:}s^3{:}s^4{:}st{:}s^2t{:}s^3t{:}s^2t^2)$, *which has a double point at* $(0{:}\ldots{:}0{:}1)$.

For the proof, which is beyond the scope of this paper, we refer to [12] or [18].

Theorem 17. *Every sequence of successive unprojections terminates.*

If the sequence contains a nonsingular surface, then all subsequent unprojections are also nonsingular, because we cannot get rid of double points by projection. Then it is also clear that the sequence terminates, because by unprojecting nonsingular Del Pezzo surfaces we cannot create new lines (as lines always project to lines), but we will erase at least one line. This follows from the fact that all lines not passing through the center of projection are also there on the image of projection. If the image is nonsingular, then there is no line passing through the center, because such a line would project to a double point (see remark 15).

Unprojection can create new lines if the exceptional line contains double points. We do not give a termination proof for this case, because this would require a deeper analysis of the type of double points of Del Pezzo surfaces. For a full proof of termination (using a different approach), we refer to [18].

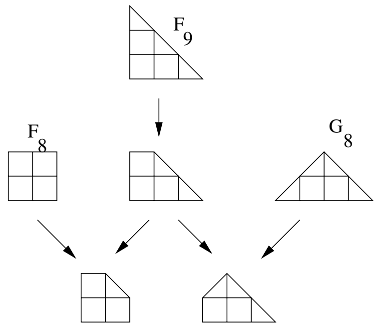

Fig. 4. A piece of the graph of Del Pezzo surfaces

Corollary 18. *The graph of Del Pezzo surfaces is connected. For every Del Pezzo surface except F_8 and G_8, there is a sequence of unprojections terminating with F_9.*

Proof. To show that the graph is connected, it suffices to show that there are paths of projections and unprojections connecting F_9, F_8, and G_8. These paths are shown in fig. 4. The lattice polygons represent monomial parametrizations of Del Pezzo surfaces (see also remark 6).

The surfaces F_8 and G_8 have a group of projective automorphisms that acts transitively on the nonsingular points. Therefore, there is, up to projective isomorphism, just one projection image of F_8 and one from G_8, namely those depicted in fig. 4.

Let S be a Del Pezzo surface different from F_8 and G_8. Then there is a sequence of unprojections terminating with F_9, F_8, or G_8. If it terminates with F_8 or G_8, then the last but one surface can also be unprojected to F_9, as can be seen in fig. 4.

Corollary 19. *Every Del Pezzo surface $S \subset \mathbb{P}^n$ except F_8 and G_8 has a parametrization by cubic polynomials through $9 - n$ base points (infinitely near base points are allowed).*

Proof. Every such Del Pezzo surface is a projection from S_9, which is parametrized by cubic polynomials. If S has degree n, then we need n projection steps, each introducing one base point.

Here is an algorithm to construct a parametrization of an implicitly given Del Pezzo surface. It assumes that we have given a Del Pezzo surface $S \subset \mathbb{P}^n$ in implicit form.

1. Reduce to the case $n = 4$ by some projection or unprojection steps.
2. Compute a line on S.
3. Project from the line. This is a birational map onto $\mathbb{P}^2$.
4. Compute the inverse of the map.
5. Reverse the projection/unprojection steps from step 1.

Example 20. Let S be the cubic surface from example 12. We already did the unprojection to a surface $S_0 \subset \mathbb{P}^4$ with equations

$$x_3x_4 - x_1^2 - x_2^2 + x_0^2 = 3x_0x_4 + 3x_0x_3 - 10x_1x_2 = 0.$$

The surface S_0 contains the line $(3{:}3{:}3p{:}9p{:}p)$. The projection from this line is given by $(x_0{:}\ldots{:}x_4) \mapsto (x_0 - x_1 : 3x_2 - x_3 : x_2 - 3x_4)$.

We do the linear coordinate change

$$(x_0, \ldots, x_4) = (y_3, y_0 + y_3, y_2 + 3y_4, y_1 + 9y_4, y_4)$$

in order to move l to a coordinate subspace. The transformed system is

$$y_1y_4 - y_0^2 - 2y_0y_3 - y_2^2 - 6y_2y_4 = 3y_1y_3 - 10y_0y_2 - 30y_0y_4 - 10y_2y_3 = 0.$$

This is a linear system for y_3, y_4. The inverse of the projection is given by the solution to this system. The parametrization of S_0 can then easily be computed by plugging into the above change of coordinates. The parametrization of S is then computed even more easy, we just have to truncate the last coordinate function.

Remark 21. In steps 1 and 2, we need some line on the surface. This can be done by plugging the parametrization of a general line into the equations and solving for the coefficients of the general line. It pays off to first project the surface into 3-space before, because this reduces the number of unknowns.

For the inversion of a birational map, we refer to [16].

Remark 22. For nonsingular Del Pezzo surfaces of degree less than or equal to 7, the number of lines depends only on the degree (e.g. nonsingular cubic surfaces have 27 lines). The incidence graph of the lines is also determined by the degree. See [12] for details.

7 Del Pezzo Surfaces of Degree 2 and 1

In order to describe Del Pezzo surfaces of degree 2 and 1, we need to introduce a generalization of projective spaces, namely weighted projective spaces (see also the short note [14]).

Let $w := (w_0, \ldots, w_n)$ be a vector of positive integers. *Weighted projective space* $\mathbb{P}_w$ is defined as the quotient of $\mathbb{C}^{n+1} - \{0\}$ by the equivalence relation identifying $(x_0, \ldots, x_n)$ with $(\lambda^{w_0} x_0, \ldots, \lambda^{w_n} x_n)$, for any $\lambda \in \mathbb{C}^*$. This is an algebraic variety of dimension n. If $w = (1, \ldots, 1)$, then $\mathbb{P}_w$ is just $\mathbb{P}^n$.

Weighted projective varieties are algebraic subvarieties of $\mathbb{P}_w$. They are defined by *weighted homogeneous* polynomials. The weigthed degree of a monomial summand $x_0^{e_0} \ldots x_n^{e_n}$ is defined as $\sum_i w_i e_i$, and a polynomial is weighted homogeneous iff all its monomials have the same weighted degree.

For any projective variety $X \subset \mathbb{P}^n$, the Hilbert function $H : \mathbb{N} \to \mathbb{N}$ is defined by setting $H(m)$ as the dimension of the quotient vector space of all forms of degree m in $x_0, \ldots, x_n$ modulo the vanishing ideal of X. For large m, the function H is a polynomial. Its degree is the dimension of X. If the dimension is r, then the leading coefficient of the Hilbert polynomial times $r!$ is equal to the degree of X (see [9]).

Using the Hilbert function, we can define the degree also for varieties in weighted projective spaces. It is natural to define that a surface S has almost minimal degree d if the leading coefficient of the Hilbert polynomial is $\frac{d}{2}$, and the value of the Hilbert function at $m = 1$ is $d + 1$. (In the case of ordinary projective space, this is equivalent to saying that S has degree d and is contained in $\mathbb{P}^d$ but not in a linear subspace.)

When we add the restrictions that S is rational and not spoiled, we have defined weighted Del Pezzo surfaces. It turns out that we get two new types of Del Pezzo surfaces, namely those of degree 2 and those of degree 1.

By definition, a Del Pezzo surface of degree 2 is a surface $S \in \mathbb{P}_{1,1,1,2}$ defined by a polynomial F of weighted degree 4, subject to the following conditions. We can write $F(x_0, x_1, x_2, y)$ as $cy^2 + F_2(x_0, x_1, x_2)y + F_4(x_0, x_1.x_2)$ for a suitable constant c and polynomials F_2, F_4 of degree 2 and 4, and we define the quartic polynomial $D(x_0, x_1, x_2) := \mathrm{disc}_y(F) = F_2^2 - 4cF_4$.

1. The discriminant D is squarefree. (One can show that otherwise S is a spoiled surface of minimal degree.)
2. The discriminant D has no four-fold point. This just excludes 4 lines meeting in a point. (One can show that otherwise S is not rational.)

Let S be a Del Pezzo surface of degree 2. The projection onto the first three projective coordinates projects S onto $\mathbb{P}^2$. This map is actually defined everywhere (because the point (0:0:0:1) does not lie on S), and has tracing index 2. The inverse image of a line $l \subset \mathbb{P}^2$ is in general an elliptic curve on S. If l is a tangent to the discriminant curve $D = 0$, then the inverse image is rational. If l is a bitangent, i.e. l is tangent at two points, then the inverse image has two components, both of which are rational. In such a case, the two components are called *pseudo-lines*. They play a similar role as the lines of Del Pezzo surfaces in ordinary projective space.

Any pseudo-line l can be defined by a linear equation $L(x_0, x_1, x_2) = 0$ and a weighted quadratic equation of type $y - Q(x_0, x_1, x_2) = 0$. The unprojection is given as the image of S under the rational map defined by $\left(x_0, x_1, x_2, \frac{y-Q}{L}\right)$.

Example 23. Let S be the Del Pezzo surface given by the equation

$$y^2 - 10x_1x_2y + 9x_0^2x_1^2 + 9x_0^2x_2^2 - 9x_0^4 = 0$$

in $\mathbb{P}_{1,1,1,2}$. The discriminant is $100x_1^2x_2^2 - 36x_0^2x_1^2 - 36x_0^2x_2^2 + 36x_0^4$. The line $x_0 = 0$ is a bitangent. For computing the inverse image, we set x_0 to 0 and get the equation $y^2 - 10x_1x_2y$, which factors into $y(y - 10x_1x_2)$. Each of the two factors give one pseudo-line.

We use the pseudo-line $x_0 = y = 0$ for unprojection. The unprojection map is $\left(x_0 : x_1 : x_2 : \frac{y}{3x_0}\right)$, and the image is the cubic surface

$$3x_0x_1^2 + 3x_0x_2^2 + 3x_0x_3^2 - 3x_0^3 - 10x_1x_2x_3 = 0.$$

Projection from a nonsingular point of a cubic Del Pezzo surface $S_0 \subset \mathbb{P}^3$ (say the point (0:0:0:1)) is more than just omitting the last coordinate: we also need to give a value for the additional coordinate y of weight 2. This value is not uniquely determined. It is the product of x_3 with the leading coefficient of the cubic equation with respect to x_3 (which is a linear polynomial because (0:0:0:1) is a nonsingular point), plus an arbitrary quadratic form in x_0, x_1, x_2.

Theorem 24. *Let $S \subset \mathbb{P}_{1,1,1,2}$ be a Del Pezzo surface of degree 2. Then S has a parametrization with the first three coordinate functions being cubics through 7 base points, and the fourth coordinate function being a sextic vanishing doubly at the 7 base points.*

Proof. Every quartic has a bitangent. So, take one, and use one of the two pseudo-lines in the preimage for unprojection. Let S_0 be the resulting cubic Del Pezzo surface. By Corollary 19, S_0 has a parametrization $(C_0{:}C_1{:}C_2{:}C_3)$ by cubic through 6 base points $p_1, \dots, p_6$. By projection, we introduce an additional base point p_7. The first three coordinate functions C_0, C_1, C_2 (which are part of the parametrization of S) pass also through p_7. The fourth component of the parametrization of S can be computed as $F := L(C_0, C_1, C_2)C_3 + Q(C_0, C_1, C_2)$, where L is the equation of the tangent plane to the projection center (0:0:0:1), and Q is an arbitrary quadratic form. Hence F has degree 6, and vanishes doubly at $p_1, \dots, p_6$. But $L(C_0, C_1, C_2)$ vanished doubly at p_7, therefore F also has a double point at p_7.

Remark 25. A nonsingular plane quartic has exactly 28 bitangents. Because a Del Pezzo surface of degree 2 is nonsingular iff its discriminant is nonsingular, we see that the number of pseudo-lines on a nonsingular Del Pezzo surface of degree 2 is 56.

Let us now turn to Del Pezzo surfaces of degree 1. By definition, this is a surface $S \in \mathbb{P}_{1,1,2,3}$ defined by a polynomial F of weighted degree 6, subject to the following conditions. We can write $F(x_0, x_1, y, z)$ as $c_1z^2 + c_2y^3 + F_1yz + F_2y^2 + F_3z + F_4y + F_6$ for a suitable constants c_1, c_2 and polynomials F_1, F_2, F_3, F_4, F_6 in x_0, x_1 of degree $1, 2, 3, 4, 6$, and we define the polynomial $D(x_0, x_1, y) := \mathrm{disc}_z(F)$ (a weighted polynomial of degree 6).

1. The discriminant D is squarefree, and $c_2 \neq 0$. (One can show that otherwise S is a spoiled surface of minimal degree.)
2. The discriminant D has at most one triple point. (One can show: if D is squarefree, and $c_2 \neq 0$, then it has at most two triple points, and if it has two triple points, then S is not rational.)

Similar as for Del Pezzo surfaces of degree 2, chopping of the coordinate z gives a rational map of tracing index 2. The image is the weighted projective plane $\mathbb{P}_{1,1,2}$. There are two kinds of pseudo-lines. When the inverse image of a curve of weighted degree 2, not passing through the point (0:0:1), splits into two components, both of them are pseudo-lines of the first kind. The second type arises as the inverse image of a curve of weighted degree 1, if this inverse image contains a singular point.

Example 26. Let $S \subset \mathbb{P}_{1,1,2,3}$ be given by the equation

$$z^2 - y^3 - x_0^4 x_1^2 - 2x_0^3 x_1^3 - x_0^2 x_1^4 = 0.$$

The inverse image of $y = 0$ splits into two pseudo-lines $y = z \pm (x_0^2 x_1 + x_0 x_1^2) = 0$ of the first kind.

The unprojection map with respect to one of them is

$$(x_0{:}x_1{:}y{:}z) \mapsto \left(x_0{:}x_1{:}\frac{z + x_0^2 x_1 + x_0 x_1^2}{y}{:}y \right).$$

Its image is the surface in $\mathbb{P}_{1,1,1,2}$ with equation

$$(x_3 y - 2x_0^2 x_1 - 2x_0 x_1^2)x_3 - y^2 = 0.$$

Example 27. Let $S \subset \mathbb{P}_{1,1,2,3}$ be the surface in example 26. The point $p := (1{:}0{:}0{:}0)$ is a double point of S. There is a unique form of weighted degree 1 vanishing at p, namely x_1. This gives the pseudo-line $z^2 - y^3 = x_1 = 0$. Its unprojection map is

$$(x_0{:}x_1{:}y{:}z) \mapsto \left(x_0{:}x_1{:}\frac{y}{x_1}{:}\frac{z}{x_1} \right),$$

and the equation of the image is

$$y^2 - x_2^3 x_1 - x_0^4 - 2x_0^3 x_1 - x_0^2 x_1^2 = 0.$$

Theorem 28. *Let $S \subset \mathbb{P}_{1,1,2,3}$ be a Del Pezzo surface of degree 1. Then S has a parametrization with the first two coordinate functions being cubics through 8 base points, the third coordinate function being a sextic vanishing doubly at the 7 base points, and the fourth coordinate function being a ninetic vanishing triply at the 7 base points.*

The proof is similar to the proof of Theorem 24.

Remark 29. The number of pseudo-lines on a nonsingular Del Pezzo surface of degree 1 is 240. See [12] for a proof.

The parametrization algorithm in sect. 6 can easily be generalized to Del Pezzo surfaces of degree 2 and 1. The so constructed parametrizations are of the type described in the theorems 24 and 28.

8 Real Del Pezzo Surfaces

If the system of equations defining a complex Del Pezzo surface are real numbers, then set of real solutions – if not empty – form a real algebraic surface, which we call a *real Del Pezzo surface*.

Projection from real nonsingular points and unprojection using real lines (or pseudo-lines in degree 2 or 1) works exactly as in the complex case. A new construction is the projection from a pair (p, p') of complex conjugate points. Both points must be nonsingular, and not lying on a common line on S. The result is over the complex numbers isomorphic to the result of two subsequent projections. The result can be realized as a real algebraic surface, because it is the projection from the line pp', and this is a real line.

Similarily, we have a new construction of unprojection using a pair of complex conjugate lines (or pseudo-lines). The two lines must not meet in a nonsingular point, because otherwise unprojection from one line would delete the other line.

Projection and unprojection are real birational maps. The number of connected components is invariant under real birational maps. But this number is not always the same for all real Del Pezzo surfaces. For instance, there are cubics with one component and cubics with two component. Other examples are given below. Therefore, the real graph of Del Pezzo surfaces is not connected.

Here is the classification of maximal vertices of this graph. The proof is again beyond the scope of this paper; we refer to [20].

Theorem 30. *Let S be a real Del Pezzo surface without a real (pseudo-)line and without a pair of complex conjugate (pseudo-)lines that do not intersect each other. Then S is one of the following.*

1. *one of the surfaces F_9, F_8, or G_8, appearing in the complex classification Theorem 16. All these surfaces have one component;*
2. *a surface in $\mathbb{P}^8$ with parametrization $(1{:}s{:}s^2{:}t{:}st{:}(s^2+t^2)s{:}t^2{:}(s^2+t^2)t{:}(s^2+t^2)^2)$, which has one component;*
3. *a Del Pezzo surface of degree 4 with two components;*
4. *a Del Pezzo surface of degree 2 with three or four components;*
5. *a Del Pezzo surface of degree 1 with five components.*

Remark 31. It is easy to see that surface 2 in the above classification is isomorphic to F_8 over the complex numbers. Over the reals, they are not isomorphic. In order to see this, note that F_8 has two one-parameter-families of conics, setting either s or t to a constant parameter. But surface 2 has no real conic at all.

Example 32. Let $S \subset \mathbb{P}^4$ be the Del Pezzo surface

$$x_1^2 + x_2^2 - x_0^2 = x_3^2 + x_4^2 - x_1x_2 = 0.$$

There are no real lines on S, and 8 complex lines. These are the lines p_iq_j, $i = 1, \ldots, 4$, $j = 1, 2$, where p_1, p_2, p_3, p_4 are the four real intersection points of the conic C : $x_1^2 + x_2^2 - x_0^2 = x_3 = x_4 = 0$ with the hyperplanes $x_1 = 0$ and $x_2 = 0$, and q_1, q_2

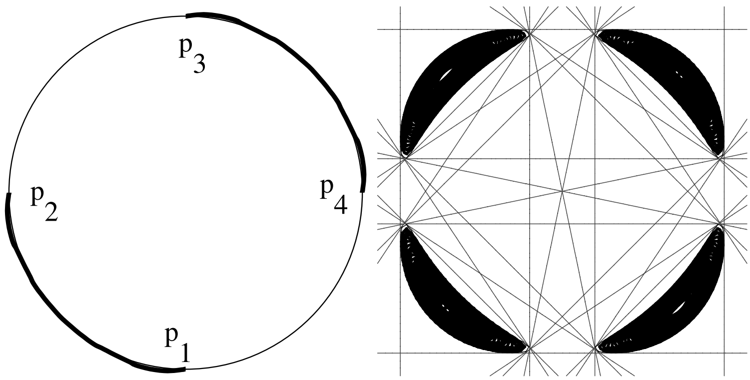

Fig. 5. *Left*: Planar picture of a Del Pezzo surface with 2 components. *Right*: A quartic curve with 28 real bitangents.

are the conjugate complex points $(0 : 0 : 0 : 1 : \pm i)$. For each $i = 1, \ldots, 4$, the line $p_i q_1$ is conjugate to the line $p_i q_2$, and this pair of conjugates meets in p_i. Since p_i is a nonsingular point of S (the only singularities on S are q_1 and q_2), unprojection is not possible.

In order to see the two connected components, we project the surface onto the first three projective coordinates. The complex image is the conic C. The real image is the subset of points on the conic for which the form $x_1 x_2$ is positive or zero. This subset of conics has two components, namely the arc connecting $p_1 p_2$ and the arc $p_3 p_4$ in the notation as in fig. 5 (*left*).

Example 33. Let $F(x_0, x_1, x_2)$ be the quartic equation

$$F = 17(x_1^4 + x_2^4) + 30x_1^2 x_2^2 - 160(x_1^2 + x_2^2)x_0^2 + 380x_0^4,$$

and let S be the Del Pezzo surface with equation $y^2 + F$ in $\mathbb{P}_{1,1,1,2}$. The quartic F has 28 real bitangents (see fig. 5, right). Each preimage splits into pair of complex conjugate pseudo-lines, intersecting each other in two real points on the surface, namely the preimages of the tangential points. Therefore unprojection is not possible.

The surface has 4 components, each projecting onto one of the four components of the subset of the plane defined by $F \leq 0$ (the black regions in fig. 5, right).

Example 34. An example of a Del Pezzo surface with 5 components is given by the equation

$$x_0^6 + x_1^6 + 2(x_0^4 + x_1^4)y - 0.9x_0^2 x_1^2 y - y^3 + z^2 = 0$$

in $\mathbb{P}_{1,1,2,3}$.

Since a real parameterization of tracing index 1 is a real birational map, only the surfaces with one component may have such a parametrization. For all maximal vertices in the real graph of Del Pezzo surfaces that have only one component, we have given a parametrization in Theorem 30. It follows that a real Del Pezzo surface has a birational parametrization if and only if it has one component.

We can say a bit more: projection does not increase the degree of the parametrization; and for the maximal vertices, we have given parametrizations of degree 3 and 4. It follows that every real Del Pezzo surface with one component has a birational parametrization of degree 3 or 4.

For real Del Pezzo surfaces with two, three, or four components, one can construct parametrizations which are not birational. No such construction is known for Del Pezzo surfaces with five components. In particular, we do not know if the surface in example 34 has a real parametrization or not.

Example 35. In order to construct a parametrization of the surface S in example 32, we first give a parametrization of the arc p_1p_2 of the conic C:

$$(x_0{:}x_1{:}x_2) = \left(1{:}\frac{2(t^2+1)}{(t^2+1)^2+1}{:}\frac{(t^2+1)^2-1}{(t^2+1)^2+1}\right),$$

by composing a well-known parametrization of the conic with the function $t \mapsto t^2+1$. This is an algebraic way of restricting the parameter space to the interval $[1,\infty)$.

This parametrization is plugged into the equation $x_3^2+x_4^2-x_1x_2$, leaving the problem of parametrizing a circle with radius $\frac{2t^2(t^2+1)(t^2+2)}{((t^2+1)^2+1)^2}$. Such a parametrization can be computed by a projection from the point $\left(\frac{t(\sqrt{2}t^2-2)}{t^2+2t+2}, \frac{(\sqrt{2}+2)t^2}{t^2+2t+2}\right)$ to a line followed by an unprojection:

$$(x_0{:}x_3{:}x_4) = \left(1{:}\frac{t(-\sqrt{2}t^2+2-4ts-2ts\sqrt{2}+s^2\sqrt{2}t^2-2s^2)}{(t^4+2t^2+2)(1+s^2)}{:}\frac{-t(-2t-\sqrt{2}t+2s\sqrt{2}t^2-4s+2ts^2+ts^2\sqrt{2})}{(t^4+2t^2+2)(1+s^2)}\right)$$

The computation was done with the help of the computer algebra system Maple. Concatenation of these two parametrizations gives a parametrization of S with tracing index 2.

Remark 36. The technique used in example 35 can be used to parametrize arbitrary Del Pezzo surfaces of degree 4 with two components (and therefore all Del Pezzo surfaces with two components, because we can reduce to degree 4 by unprojection): compute a projection with conic fibers, restrict the parameter space algebraically, parametrize the parametric family of conics. A similar technique can also be used for Del Pezzo surfaces with 3 components (see also [13, 17]).

In order to parametrize Del Pezzo surfaces with 4 components, it is theoretically possible to use the construction in [19] which works over arbitrary fields. Unfortunately, the so constructed parametrization has tracing index 24 and is very complicated. The author computed a parametrization of example 33 with this method, but the output fills several pages.

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