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Rational and Nearly Rational Varieties

J. KOLLÁR, K. E. SMITH AND A. CORTI

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RATIONAL AND NEARLY RATIONAL VARIETIES

The most basic algebraic varieties are the projective spaces, and rational varieties are their closest relatives. In many applications where algebraic varieties appear in mathematics and the sciences, we see rational ones emerging as the most interesting examples. The authors have given an elementary treatment of rationality questions using a mix of classical and modern methods. Arising from a summer school course taught by János Kollár, this book develops the modern theory of rational and nearly rational varieties at a level that will particularly suit graduate students. There are numerous examples and exercises, all of which are accompanied by fully worked out solutions, that will make this book ideal as the basis of a graduate course. It will act as a valuable reference for researchers whilst helping graduate students to reach the point where they can begin to tackle contemporary research problems.

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AND
ALESSIO CORTI

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Introduction

The most basic algebraic varieties are the projective spaces, and rational varieties are their closest relatives. Rational varieties are those that are birationally equivalent to projective space. In many applications where algebraic varieties appear in mathematics, we see rational ones emerging as the most interesting examples. This happens in such diverse fields as the study of Lie groups and their representations, in the theory of Diophantine equations, and in computer-aided geometric design.

This book provides an introduction to the fascinating topic of rational, and "nearly rational," varieties. The subject has two very different aspects, and we treat them both. On the one hand, the internal geometry of rational and nearly rational varieties tends to be very rich. Their study is full of intricate constructions and surprising coincidences, many of which were thoroughly explored by the classical masters of the subject. On the other hand, to show that particular varieties are *not* rational can be a difficult problem: the classical literature is riddled with serious errors and gaps that require sophisticated general methods to repair. Indeed, only recently, with the advent of minimal model theory, have all the difficulties in classical approaches to proving nonrationality based on the study of linear systems and their singularities been ironed out.

While presenting some of the beautiful classical discoveries about the geometry of rational varieties, we pay careful attention to arithmetic issues. For example, we consider whether a variety defined over the rational numbers is rational *over* $\mathbb{Q}$, which is to say, whether there is a birational map to projective space given locally by polynomials with coefficients in $\mathbb{Q}$.

The hardest parts of the book focus on how to establish nonrationality of varieties, a difficult problem with many basic questions remaining open today. There are good general criteria, involving global differential forms, that can be used in many cases, but the situation becomes very difficult when these tests fail. For example, using simple numerical invariants called the plurigenera, it is easy to see that a smooth hypersurface in projective space whose degree exceeds its embedding dimension can not be rational. However, it is a very delicate problem to determine whether or not a lower degree hypersurface is rational.

Rationality of quadric and cubic *surfaces* was completely settled in the nineteenth century, but rationality for threefolds occupied the attention of algebraic geometers for most of the twentieth century. In the 1970s, Clemens and Griffith identified a new obstruction to rationality for a threefold inside its third topological (singular) cohomology group. This method of *intermediate Jacobians* provided the first proof that no smooth cubic threefold is rational. Because this approach fits better in a book about Hodge theory, we do not discuss it here. Instead, we prove that no smooth quartic threefold in projective four-space is rational, drawing on ideas from the minimal model program. Beyond this, very little is known: no one knows whether or not all smooth cubic fourfolds are rational, or indeed, whether there exists any nonrational smooth cubic hypersurface of any dimension greater than three.

On the other hand, in this book we do present a technique for proving nonrationality of "very general" hypersurfaces of any dimension greater than two whose degree is close to their dimension. Like other approaches to proving nonrationality, this technique uses differential forms; the novelty here is that the differential forms we use are defined on varieties of prime characteristic.

Our biggest omission is perhaps never to define precisely what we mean by a "nearly rational variety." Current research in birational algebraic geometry indicates that the most natural class of nearly rational varieties is formed by *rationally connected varieties*, introduced in Kollár *et al*. (1992). Although it is easy to state the definition, it is harder to appreciate why we claim that this is indeed the most natural class of nearly rational varieties to consider. Our aim in this book is more modest; we hope to inspire the reader to learn more about rationality questions. As a next step, we recommend the general introduction to rationally connected varieties given in Kollár (2001). Kollár (1996) contains a detailed treatment for the technically advanced.

Description of the chapters

Chapter 1 describes some basic examples of rational varieties, concentrating on quadric and cubic hypersurfaces. We give fairly complete answers for quadric hypersurfaces, but many open questions remain about cubics. We also discuss the simplest nonrationality criteria in terms of differential forms.

Cubic surfaces are examined in detail in Chapter 2. This is a classical topic that began with the works of Schläfli and Clebsch and culminated with the arithmetic studies of Segre and Manin. The main results here are about smooth cubic surfaces of Picard number one: no such cubic surface is rational. This is essentially an arithmetic result, since cubic surfaces over an algebraically closed field never have Picard number one and are always rational. On the other hand, the techniques are quite geometric, and show many of the higher dimensional methods in simpler form.

A general study of rational surfaces is given in Chapter 3. For instance, we prove Castelnuovo's criterion for rationality, giving a simple numerical characterization of rationality for smooth complex surfaces. Although this result is classical, we develop it within the modern framework of the minimal model program. This allows us to also treat surfaces that are not defined over an algebraically closed field.

In Chapter 4, we construct examples of higher dimensional smooth nonrational hypersurfaces of low degree. The constructed varieties are all Fano, which means in particular that the naive numerical invariants introduced in Chapter 1 all vanish here even though the varieties are not rational. Our proof is based on the method of reduction to prime characteristic, where we are able to exploit some of the quirks of differential forms arising from the peculiarity that the derivative of a pth power is zero in characteristic p. These positive characteristic varieties are then lifted to get examples over $\mathbb{C}$. While this method yields many examples of smooth nonrational varieties, it is not capable of producing complete families such that every smooth member is nonrational.

Chapter 5 develops the Noether–Fano method, a technique for proving nonrationality of higher dimensional varieties, analogous to the ideas presented in Chapter 2 to treat cubic surfaces. Using this approach, we produce complete families of Fano varieties in which no smooth member is rational. This example, presented in Section 5.3, is by far the simplest higher dimensional application of the Noether–Fano method. We also start the proof that no smooth quartic threefold in projective four-space is rational. This fact about quartic threefolds was first claimed by Fano (1915) although a complete proof appeared only later with the work of Iskovskih and Manin (1971).

In Chapter 6, we present more advanced machinery, namely the theory of singularities of pairs, for carrying out the general method developed in Chapter 5. Our main application is the proof of a particular numerical result which is a key ingredient in the proof that no quartic threefold is rational. These techniques also have numerous applications to diverse problems of higher dimensional geometry.

Chapter 7 contains the solutions of the exercises. The reader is strongly urged to try to work them out first instead of going to the solutions straight away.

This book began with a series of lectures by J. Kollár given at the European Mathematical Society Summer School in Algebraic Geometry in Eger, Hungary in August 1996. The notes were written up by K. E. Smith. Later new chapters were added and the old ones have been revised and reorganized. Section 4.7 (by J. Rosenberg) answers a problem raised in the original lectures.

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Prerequisites

We have devoted considerable effort to making our exposition as elementary as possible. Chapters 1 and 2 should be accessible to students who have completed a year long introductory course on classical algebraic geometry, for instance along the lines of Shafarevich (1994, vol.1). In particular, we use the language of linear systems of curves on surfaces, including their intersection theory, but we do not use cohomology.

In Chapter 3, we use basic facts about intersection theory on surfaces and cohomology for line bundles on curves and surfaces, including the Riemann–Roch theorem, Serre duality, the adjunction formula, and the Kodaira vanishing theorem. We use the most rudimentary aspects of the theory of schemes of finite type over a field in our discussion of the field of definition of a variety. Hodge theory is also mentioned in a peripheral way. Reid's lectures (1997) are an excellent and concise summary of much of the material needed in Chapter 3 and later in the book. Students familiar with Sections IV and V of Hartshorne's book (1977) should be more than adequately prepared for Chapter 3.

In Chapter 4, we work with schemes over Spec $\mathbb{Z}$ and their Kähler differentials, but we carefully explain all that is used beyond the most basic definitions. We hope that this chapter will help those familiar with classical algebraic geometry to appreciate the theory of schemes.

Chapters 5 and especially 6 are somewhat harder. We assume more sophistication in manipulating $\mathbb{Q}$-divisors, and use two major theorems that the reader is asked to accept without proof, namely the Lefschetz theorem on the Picard group of hypersurfaces and Hironaka's results on the resolution of singularities. One technically more demanding proof (of Theorem 6.32) is relegated to an Appendix. Chapter 6 may be the hardest, mainly because of the number of new concepts involved. It serves as a good introduction to some more advanced books on birational geometry or the minimal model program, for instance to Kollár and Mori (1998).

Notation and basic conventions

Let k be a field. Our main interest is in the cases $k = \mathbb{C}$ for studying geometric properties and $k = \mathbb{Q}$ for investigating the arithmetical questions. We occasionally encounter other cases too, for instance the finite fields $\mathbb{F}_q$, the real numbers $\mathbb{R}$, p-adic fields $\mathbb{Q}_p$ and algebraic number fields. The algebraic closure of k is denoted $\bar{k}$.

The notation for the ground field is suppressed when the field is clear from the context or irrelevant, but occasionally we write X_k to emphasize that the variety X is defined over the ground field k. If $L \supset k$ is a field extension then X_L denotes the variety X_k viewed as being over L. Technically speaking, $X_L = X_k \times_{\operatorname{Spec} k} \operatorname{Spec} L$.

If L is any field containing k, then an *L-point*, or an *L-rational point*, is one having all of its coordinates defined over L. That is, thinking of a variety over k as locally a subvariety of $\mathbb{A}^n$ given by the vanishing of polynomials with coefficients in k, then an L-rational point is given by an n-tuple of elements of L satisfying the defining polynomials. Thinking more scheme-theoretically, an L-point on a scheme X can be defined as a morphism $\operatorname{Spec} L \to X$. In particular, a k-rational point on a k-scheme corresponds to a maximal ideal whose residue field is k. The symbol $X(L)$ denotes the set of L-points of X.

Morphisms and *rational maps* between varieties are always assumed to be defined over the ground field, except where explicitly stated otherwise. Likewise, linear systems on a variety X_k are assumed defined over k.

Morphisms are denoted by solid arrows $\to$ and rational maps by dashed arrows $\dashrightarrow$. The "image" of a rational map is the closure of the image of the morphism obtained by restricting the rational map to some nonempty open set where it is defined; in the same way, we define the image of a subvariety under a rational map, provided that the map is defined at its generic point. In particular, let $f : Y \dashrightarrow X$ be a rational map and suppose that f is defined at the generic point of some subvariety Z of Y. Then the image of Z on X, denoted $f_*(Z)$, is the closure in X of the set $f|_{X_0}(Z \cap X_0)$, where X_0 is some open set meeting Z on which f is a well-defined morphism. In the case of birational maps, the image is also called the *birational transform*, especially in the case where this image has the same dimension.

Let X be a normal variety. An irreducible and reduced subscheme of codimension one is called a *prime divisor*. A *divisor* on X is a formal linear combination $D = \sum d_i D_i$ of prime divisors where $d_i \in \mathbb{Z}$. In using this notation we assume that the D_i are distinct. A *$\mathbb{Q}$-divisor* is a formal linear combination $D = \sum d_i D_i$ of prime divisors where $d_i \in \mathbb{Q}$. The divisor D is called *effective* if $d_i \geq 0$ for every i. A divisor (or $\mathbb{Q}$-divisor) D is called *$\mathbb{Q}$-Cartier* if mD is

Cartier for some nonzero integer m, where by Cartier we mean that it is locally defined by a single equation. On a smooth variety every divisor is Cartier. The *support* of $D = \sum d_i D_i$, denoted by Supp D is the subscheme $\cup_{d_i \neq 0} D_i$. *Linear equivalence* of two divisors is denoted by $D_1 \sim D_2$.

A property of a variety X_k refers to the variety considered over k. We add the adjective *geometrically* when talking about a property of $X_{\bar{k}}$. For example, the affine plane curve defined by the equation $x^2 + y^2 = 0$ is irreducible as a $\mathbb{Q}$-variety but it is geometrically reducible. For many properties (including smoothness or projectivity), the distinction does not matter.

Varieties are assumed reduced and irreducible, except where explicitly stated otherwise. In particular, the terms "smooth curve" and "smooth surface" always refer to *connected* smooth surfaces and curves. The one exception is that we use the term "curve on a surface" to mean any effective divisor, which may or may not be reduced and irreducible. Because we are concerned with birational properties, there is no loss of generality in assuming all varieties to be quasi-projective. In any case, our main interest is in smooth projective varieties.

In writing these notes, our policy was not to be sidetracked by anomalies in positive characteristic. These usually appear when the base field is not perfect, that is, when it has algebraic extensions obtained by taking pth roots (in characteristic p). Technical problems related to such issues are relegated to exercises and they can be safely ignored for most of the book.

The exception is Chapter 4 where the unusual properties of such field extensions are exploited to prove several results about varieties over $\mathbb{C}$ or $\mathbb{Q}$.

1
Examples of rational varieties

In this chapter, we introduce rational varieties through examples. After giving the fundamental definitions in the first section and settling the rationality question for curves in Section 2, we continue with the rich theory of quadric hypersurfaces in Section 3. This is essentially a special case of the theory of quadratic forms, though the questions tend to be strikingly different.

Quadrics over finite fields are discussed in Section 4. Several far-reaching methods of algebraic geometry appear here in their simplest form.

Cubic hypersurfaces are much more subtle. In Section 5, we discuss only the most basic rationality and unirationality facts for cubics. A further smattering of rational varieties is presented in Section 6, together with a more detailed look at determinantal representations for cubic surfaces.

A very general and useful nonrationality criterion, using differential forms, is discussed in Section 7.

1.1 Rational and unirational varieties

Roughly speaking, a variety is *unirational* if a dense open subset is parametrized by projective space, and *rational* if such a parametrization is one-to-one.

To be precise, fix a ground field k, and let X be a variety defined over k. It is important to bear in mind that k need not be algebraically closed and that all constructions involving the variety X are carried out over the ground field k.

DEFINITION 1.1. A variety is *rational* if it is birationally equivalent to projective space. Explicitly, the variety X is rational if there exists a birational map $\mathbb{P}^n \dashrightarrow X$.

DEFINITION 1.2. The variety X is *unirational* if there exists a generically finite dominant map $\mathbb{P}^n \dashrightarrow X$.

Rational varieties were once called “birational,” in reference to the rational maps between them and projective space in each direction. “Unirationality” thus refers to the map from $\mathbb{P}^n$ to the variety, defined in one direction only. This explains the odd use of the prefix “uni” in referring to a map which is finite-to-one.

We emphasize that in both definitions above, the varieties and the maps are defined over the fixed ground field k. This means that the variety X is defined locally by polynomials with coefficients in k, and also that the map can be described by polynomials with coefficients in k.

Our guiding question throughout this book is the following: *Which varieties are rational or unirational*?

The rationality or unirationality of a variety may depend subtly on the field of definition. For example, a variety defined over $\mathbb{Q}$ may be considered as a variety defined over $\mathbb{R}$. It is possible that there is a birational map given by polynomials with *real* coefficients from projective space to the variety, but there is no such birational map given by polynomials with *rational* coefficients. Our first example nicely illustrates this point.

EXAMPLE 1.3. Consider the plane conic C defined by the homogeneous equation $x^2 + y^2 = pz^2$, where p is a prime number congruent to -1 modulo 4. We claim that

1. the Diophantine equation $x^2 + y^2 = pz^2$ has no rational solutions (aside from the trivial solution $x = y = z = 0$),
2. the curve C is not rational over $\mathbb{Q}$, and
3. the curve C is rational over $\mathbb{Q}(\sqrt{p})$.

Indeed, assume that $x^2 + y^2 = pz^2$ has a rational solution. By clearing denominators, we may assume that x, y, and z are integers, not all divisible by p. If neither x nor y is divisible by p, then the congruence $x^2 \equiv -y^2 \mod p$ leads to a solution of $u^2 \equiv -1 \mod p$. But this is impossible since $p \equiv -1 \mod 4$. (This easy fact is sometimes called Euler's criterion for quadratic congruences; if you have not seen it before, check by hand the examples $p = 3, 7, 11$ before looking it up in any elementary number theory book.) This contradiction forces p to divide both x and y. But then p^2 divides pz^2, so that p divides z as well, a contradiction. This establishes (1).

Now, if C is rational (or even unirational) over $\mathbb{Q}$, then images of the rational points under the map $\mathbb{P}^1 \dashrightarrow C$ give plenty of rational points on C, contradicting (1).

Finally, (3) can be seen from the explicit parametrization

$$\mathbb{P}^1 \dashrightarrow C \subset \mathbb{P}^2$$

given by

$$(t : 1) \mapsto (t^2 - 1 : 2t : \frac{1}{\sqrt{p}}(t^2 + 1)).$$

This is a special case of the parametrization given later in the proof of Theorem 1.11 for a general quadric hypersurface.

We say that X is *geometrically rational* if X is rational over $\bar{k}$. The reader is cautioned however, that the literature is inconsistent: some authors use the term "rational" to mean "geometrically rational."

One must be careful about trusting intuition based on extensive study of algebraic varieties over an algebraically closed field. For example, even when a map $\mathbb{P}^n \dashrightarrow X$ as in Definition 1.2 is dominant, the induced map on the set of k-points $\mathbb{P}^n(k) \dashrightarrow X(k)$ can be very far from surjective. For instance, with the ground field fixed to be $\mathbb{Q}$, consider the map $\mathbb{P}^1 \to \mathbb{P}^1$ given by $(s : t) \mapsto (s^2 : t^2)$. The image of the rational points is a very sparse subset of the set of all rational points of the target variety. This is typical for maps defined over algebraically non-closed fields.

1.2 Rational curves

Over $\mathbb{C}$, and more generally, over any algebraically closed field, the only smooth projective curve remotely resembling the projective line is $\mathbb{P}^1$ itself. Indeed, as is frequently covered in a first course in algebraic geometry, the following are equivalent for a smooth projective curve over an algebraically closed field:

1. the curve is isomorphic to the projective line;
2. the curve is birationally equivalent to the projective line;
3. there is a nonconstant map from the projective line to the curve;
4. the curve has no nonzero global holomorphic (that is, Kähler) one-forms: in other words, the canonical linear system is empty.

But what about curves over algebraically non-closed fields? It is still the case that every rational map from a curve is, in fact, an everywhere-defined morphism; the usual proof of this fact does not require an algebraically closed ground field. So over any ground field, a birational map from a curve is an isomorphism, and (1) and (2) are equivalent. In this section, we see also that (3) is equivalent to (1) and to (2) over an arbitrary ground field, but that (4) is not. In fact, we see that rationality questions for curves come down to the case of plane conics, where the answers depend on the ground field.

Given a smooth projective curve, how can we tell if it is rational? Of course, if a curve is rational over k, it is certainly rational over its algebraic closure $\bar{k}$, so we might as well restrict our attention to geometrically rational curves. Among all the different representations of a smooth geometrically rational curve (for instance, as a projective line, a plane conic, a twisted cubic, and so on), the following proposition shows that the plane conics account for all possible birational models of the projective line over any field.

Proposition 1.4. *A smooth projective geometrically rational curve is isomorphic to a smooth plane conic.*

Again we emphasize (we will soon stop!) that the interesting part of this statement is that all this is going on over some fixed ground field k, which need not be algebraically closed. So any smooth curve over k that is rational when considered as a variety over $\bar{k}$ must be isomorphic (over k) to a curve in $\mathbb{P}^2$ defined by a quadratic polynomial with coefficients in the ground field k. This would be obvious if k were algebraically closed.

The proof uses two basic results of algebraic geometry over algebraically nonclosed fields. Both are quite elementary but they do not always receive the emphasis that they deserve in introductory texts.

Proposition 1.5. *Let X be a smooth quasi-projective variety defined over a field k. Then it has a canonical divisor defined over k. Thus we can speak of the canonical divisor class K_X as a linear equivalence class defined over k.*

Proof. Let us start with the most classical case when k has characteristic zero and X is a curve.

If g is any function on X, the divisor of dg is a canonical divisor. If g is in $k(X)$ then the corresponding divisor (dg) is defined over k.

We do something similar in higher dimensions. Choose $g_1, \ldots, g_n$ algebraically independent functions of $k(X)$. Then the divisor of $dg_1 \wedge \cdots \wedge dg_n$ is a canonical divisor defined over k.

We have to be a little more careful in positive characteristic. The problem is that if g is a pth power then $dg = 0$ and its divisor (dg) is not defined. It is not hard to show that this problem can be avoided by a careful choice of the functions g_i. See, for instance, van der Waerden (1991, 19.7).

Another possibility, more in keeping with modern techniques, is to construct the sheaf of differential forms (i.e. the sheaf of Kähler differential one-forms) as in Shafarevich (1994, III.5) and define the canonical class as the divisor class corresponding to its determinant bundle. □

Proposition 1.6. *Let D be a divisor on a smooth projective variety X defined over a field k. Then the dimension of the complete linear system defined* *by D does not depend on k. That is, if $K \supset k$ is any field extension then* $\dim |D|$ *is the same whether computed over k or K.*

Thus two divisors D_1, D_2, both defined over a field k, are linearly equivalent over k if and only if they are linearly equivalent over K.

PROOF. The classical argument, identifying the linear system $|D|$ with a projective space of functions f such that $(f) + D \geq 0$ is explained in Shafarevich (1994, III.3.5). The last part follows by noting that D_1 and D_2 are linearly equivalent if and only if $\dim |D_1 - D_2| = \dim |D_2 - D_1| = 0$.

Those who are familiar with the sheaf theoretic viewpoint should also look at Exercise 3.34. □

PROOF OF PROPOSITION 1.4. Let C be a smooth geometrically rational curve and let $\mathcal{O}_C(K_C)$ denote its canonical line bundle, which is defined over k by Proposition 1.5.

Because the curve is isomorphic to $\mathbb{P}^1$ over $\bar{k}$, we have isomorphisms $\mathcal{O}_C(K_C) \cong \mathcal{O}_{\mathbb{P}^1}(-2)$ over $\bar{k}$. The global sections of the dual bundle $\mathcal{O}_C(-K_C) \cong \mathcal{O}_{\mathbb{P}^1}(2)$ form a three dimensional vector space over $\bar{k}$. By Proposition 1.6, the space of global sections of $\mathcal{O}_C(-K_C)$ is also three dimensional when considered over k. These global sections define an embedding (given by the complete linear system $|-K_C|$) of C, over k, as a conic in the projective plane. □

Proposition 1.4 ensures that rationality questions for any geometrically rational curve reduce to questions about plane conics. So given a plane conic, how can we tell whether or not it is birationally equivalent to the projective line? Because every birational map from a smooth curve is actually an everywhere-defined morphism, we are essentially asking when a plane conic is isomorphic to $\mathbb{P}^1$. This, in turn, is essentially equivalent to the theory of quadratic forms in three variables. The slight difference is that in quadratic form theory one usually does not consider the forms that differ by a scalar multiple to be equivalent, although they determine the same conic.

PROPOSITION 1.7. *The following are equivalent for any smooth projective geometrically rational curve over a field k:*

1. *the curve is isomorphic to the projective line;*
2. *the curve admits a k-point;*
3. *the curve has a point defined over some odd degree field extension of k;*
4. *there is an odd degree line bundle on the curve defined over k.*

PROOF. It is clear that (1) implies (2) and that (2) implies (3).

To show that (3) implies (4), assume (3) and pick a k'-point P on the geometrically rational curve C, where k' is an extension of k of odd degree d. If k'

is separable over k, then P has d distinct conjugates $P_1 = P, P_2, \dots, P_d$ under the corresponding Galois action on C. Their union $\{P_1, \dots, P_d\}$ is defined over k (as shown in Exercise 1.8 below). Thus $\mathcal{O}_C(P_1 + \cdots + P_d)$ is a degree d line bundle defined over k, proving (4). Exercise 1.9 treats the inseparable case.

Finally, assume (4) and let L be a line bundle of degree $2r + 1$. As C is geometrically rational, the line bundle $\mathcal{O}_C(K_C)$ has degree minus two, and so $L \otimes \mathcal{O}_C(rK_C)$ is a degree one line bundle defined over k. Its global sections define an isomorphism from C to the projective line. $\square$

Exercise 1.8. Let k'/k be a finite Galois extension. The Galois group $\mathrm{Gal}(k'/k)$ acts on $\mathbb{A}^n_{k'} = \operatorname{Spec} k'[x_1, \dots, x_n]$ coordinatewise. Let X be a closed algebraic subset of $\mathbb{A}^n_{k'}$. Prove that the following are equivalent:

1. the set X can be defined by polynomials in $k[x_1, \dots, x_n]$;
2. the set X is invariant under the $\mathrm{Gal}(k'/k)$-action.

(A more advanced version of this result is discussed in Section 3.4)

Exercise 1.8 indicates why we occasionally assume that our varieties are defined over a perfect ground field in treating rationality questions. A perfect field is one that admits no inseparable extensions; in particular, the splitting field of any polynomial over a perfect field is a Galois extension. However, over a nonperfect ground field k (necessarily of prime characteristic p), there are subsets of $\mathbb{A}^n_{k'}$ that are not defined over k even though they are fixed by the group of automorphisms of k' over k.

Exercise 1.9. Let k'/k be a purely inseparable extension of degree p^a. Let C be a smooth curve in $\mathbb{A}^n_{k'}$ and let P be a k'-point of C. Prove that the divisor $p^a P$ can be defined by polynomials in $k[x_1, \dots, x_n]$.

Finally, we point out that for curves there is no difference between rationality and unirationality, regardless of whether or not the ground field is algebraically closed.

Proposition 1.10 (Lüroth theorem). *A smooth projective curve is rational if and only if there is a nonconstant map $\mathbb{P}^1 \to C$.*

Proof. If a curve C is rational, then the given birational equivalence can be taken for the needed map. For the converse, it is equivalent to show that every subfield of the function field $k(t)$ is itself purely transcendental over k. An elementary algebraic proof of this fact is given in van der Waerden (1991, 10.2).

For the more geometrically inclined reader, we show how the statement reduces easily to the case where the curve is defined over an algebraically closed ground field. Suppose that there is a nonconstant map $\mathbb{P}^1 \to C$. Assuming the statement over an algebraically closed field, this implies that C is geometrically rational. On the other hand, the image of any k-point of $\mathbb{P}^1$ produces a k-point on C, so the curve C is rational over k by Proposition 1.7. $\square$

1.3 Quadric hypersurfaces

In the previous section we proved a criterion, Proposition 1.7, for rationality of plane conics. This criterion generalizes quite nicely to quadrics of arbitrary dimension, a result due to Springer. For simplicity we assume that the characteristic of our field k is not two.

THEOREM 1.11. *The following are equivalent for any quadric hypersurface in projective space that is not the union of two hyperplanes:*

1. *the quadric is rational;*
2. *the quadric has a smooth k-point;*
3. *the quadric has a smooth k'-point for some odd degree field extension k' of k.*

PROOF. The only obvious implication is that (2) implies (3).

That (1) implies (2) seems easy but there are some pitfalls. Assume that there is a birational map $\phi : \mathbb{P}^{n-1} \dashrightarrow Q$, where Q denotes the quadric hypersurface in $\mathbb{P}^n$. Then ϕ is defined on some Zariski open subset $U \subset \mathbb{P}^{n-1}$. If k is infinite, then $U(k)$ is Zariski dense and its image is a Zariski dense set of k-points in Q. Thus we get plenty of smooth k-points on Q. This argument breaks down when k is finite, because then there are open sets with no k-points. Nonetheless, if Q is smooth, we can use Nishimura's lemma (see Exercise 1.12) below to conclude that Q has a smooth k-point. The singular case is reduced to the smooth case using Exercise 1.13 below.

Conversely, let Q be a quadric in $\mathbb{P}^n$ and let P be a smooth k-point on Q. Let $\pi : Q \dashrightarrow \mathbb{P}^{n-1}$ be the projection from Q to any hyperplane in $\mathbb{P}^n$ defined over k but not containing P. Note that π is generically one-to-one and defined over k, hence gives the desired birational equivalence. Indeed, π is one-to-one except along the lines through P lying on Q, and Q cannot be covered by such lines unless it is a cone with vertex at P. This proves that (1) follows from (2).

It remains to show that (3) implies (2). We again consider the smooth case, the singular case following from Exercise 1.13 below.

It is sufficient to consider the case when $k' = k(z)$ is a degree $d > 1$ extension generated by one element, because we can build up k' by successively adding elements. (Note that if k' is separable over k, then it is generated by one element in any case.)

Pick a k'-point P on $Q \subset \mathbb{P}^n$ and write it as

$$(a_{00} + a_{01}z + \cdots + a_{0,d-1}z^{d-1} : \cdots : a_{n0} + a_{n1}z + \cdots + a_{n,d-1}z^{d-1}),$$

where the a_{ij} are in k. Consider the map $\Phi : \mathbb{P}^1 \to \mathbb{P}^n$ sending $(s : t)$ to

$$(a_{00}s^{d-1} + a_{01}s^{d-2}t + \cdots + a_{0,d-1}t^{d-1} : \cdots : a_{n0}s^{d-1} + \cdots + a_{n,d-1}t^{d-1}).$$

Because this map is defined over k, the image is a rational curve, say C. The degree of C is at most $d-1$; let us denote this degree by d'.

If C sits on Q, then any k-point on C produces for us the desired k-point on Q. Otherwise, let $F(s,t)$ be the pullback of the equation of Q to $\mathbb{P}^1$ under Φ. Note that F has degree $2d'$, so that its dehomogenization $f(t) = F(1,t)$ also has degree $2d'$ (unless s divides F, in which case the image of $(0:1)$ under Φ is a k-point of Q). Because z is a root of f, its minimal polynomial g divides f, and f/g factors as Πh_i. Since $\deg f/g = \deg f - \deg g$ is odd, at least one of the factors, say h_1, has odd degree d'', with $d'' \leq 2d' - d \leq d - 2$. Let t_0 be a root of h_1. Then the image of the point $(1, t_0)$ under Φ is a point of Q defined over a field extension k''/k of odd degree d''. We are done by induction on the degree of the extension. □

EXERCISE 1.12. Prove Nishimura's lemma: If Y is smooth, Y' is projective, and there is a rational map $Y \dashrightarrow Y'$, then if Y has a k-point, so does Y'. Also, find a counterexample when Y is not smooth.

EXERCISE 1.13. Let Q be a quadric hypersurface in projective space defined over k. Assume that the characteristic of k is not two. Show that the singular locus of Q is a linear subspace defined over k and that Q is a cone over a smooth quadric Q' (or else Q is a double plane). Prove that Q has a smooth k-point if and only if Q' has a smooth k-point. Use this to finish the proof of Theorem 1.11.

Despite Theorem 1.11, the classification of quadrics up to birational equivalence is still not complete. Indeed, it is not clear how to decide whether a given quadric has k-points or not. The extensive theory of quadratic forms is devoted to this question and to the classification of quadrics up to isomorphism. On the other hand, there are reasonably complete answers over specific fields. In the next section, we see that the rationality problem is solved for quadrics over finite fields. For now, we mention some results for quadrics over $\mathbb{R}$ and $\mathbb{Q}$.

EXERCISE 1.14. Every quadric hypersurface Q over $\mathbb{R}$ is isomorphic to a quadric defined by the homogeneous form

$$x_1^2 + \cdots + x_p^2 - \left(x_{p+1}^2 + \cdots + x_{p+q}^2\right),$$

for some $p \geq q$. The numbers $(\dim Q, p, q)$ form a complete set of invariants for the isomorphism types of real quadric hypersurfaces. The quadric is rational if and only if $q > 0$.

The following remarkable theorem says that the rationality of quadrics over the rational numbers is essentially a real question.

THEOREM 1.15 (Hasse–Minkowski). *The following are equivalent for a smooth quadric hypersurface of dimension at least three, defined over the rational numbers $\mathbb{Q}$:*

1. *the quadric has a $\mathbb{Q}$-point;*
2. *the quadric is rational over $\mathbb{Q}$;*
3. *the quadric has an $\mathbb{R}$-point;*
4. *the quadric is rational over $\mathbb{R}$.*

Theorem 1.11 here shows the equivalence of (1) and (2), as well as (3) and (4). The surprising part is that (3) implies (1). For a proof, see, for instance, Serre (1973, IV.3).

1.4 Quadrics over finite fields

In this section, we completely settle the rationality problem for quadrics over a finite field by proving the following theorem.

THEOREM 1.16. *A quadric hypersurface over a finite field is rational, provided that it is not the union of two hyperplanes over the algebraic closure of the ground field.*

According to Theorem 1.11, in order to prove this theorem, we need only show that the quadric has a smooth point defined over the given finite ground field. In fact, by Exercise 1.13, we might as well assume that the quadric is smooth, so it is enough to prove the following theorem.

THEOREM 1.17. *A positive dimensional quadric hypersurface over a finite field has a point over that field.*

It is interesting to note that the proof of Theorem 1.17 reduces easily to the case of smooth conics in $\mathbb{P}^2$. Indeed, given any quadric hypersurface defined over k in $\mathbb{P}^{n+1}$, a hyperplane section is a quadric in $\mathbb{P}^n$. If this section is singular, then its singular set is a linear space defined over k by Exercise 1.13, and so has plenty of k-rational points. If the section is smooth, then we can look for rational points on it by induction.

We give two different proofs of the existence of such a rational point. The first, Theorem 1.18 below, is more general and elementary, and is based on some congruence statements about the number of k-rational points on varieties defined by low degree polynomials over finite fields. The second proof is more special, but its ideas lead much further, eventually to the proof of the Weil estimates for the k-rational points on any variety.

THEOREM 1.18 (Chevalley, 1935). *Let $f_1, \ldots, f_s$ be homogeneous polynomials in d variables over a finite field k. If the sum of the degrees of the f_i is less than d, then the polynomials have a nontrivial common solution in k.*

Before proving Theorem 1.18, we note that it implies Theorem 1.17 simply by considering the case of one homogeneous polynomial of degree two in at least three variables.

PROOF OF THEOREM 1.18. Consider the affine variety X defined by the polynomials $f_1, \ldots, f_s$ in $\mathbb{A}^d$. Let $q = p^m$ denote the cardinality of k, and consider the auxiliary polynomial $P := \prod_i (1 - f_i^{q-1}) \in k[x_1, \ldots, x_d]$. Because $\gamma^q = \gamma$ for all γ in k, we see that

$$P(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in X(k), \text{ and} \\ 0 & \text{if } \mathbf{x} \in k^d \setminus X(k). \end{cases}$$

Thus we obtain that

$$\sum_{\mathbf{x} \in k^d} P(\mathbf{x}) \equiv \text{number of points in } X(k) \text{ modulo } p.$$

(It is tempting to say that equality holds, but the left hand side is in k, so modulo p is the best one can claim.)

Next we compute the sum on the left. Let $\prod x_i^{a_i}$ be any monomial appearing in the polynomial P. Then

$$\sum_{\mathbf{x} \in k^d} \prod_i x_i^{a_i} = \prod_i \left(\sum_{y \in k} y^{a_i} \right).$$

By assumption, $\sum a_i \leq \deg P < d(q-1)$, hence $a_i < q - 1$ for some i. Now, for any $0 \leq a < q - 1$, there exists a (nonzero) z in k such that $z^a \neq 1$. Because

$$\sum_{y \in k} y^a = \sum_{y \in k} (zy)^a = z^a \sum_{y \in k} y^a,$$

we conclude that $\sum_{y\in k} y^a = 0$ for a in this range. Hence we conclude that $\sum_{\mathbf{x}\in k^d} \prod_i x_i^{a_i}$ is zero for each monomial $\Pi x_i^{a_i}$ occurring in P. Adding up over all such monomials, we see that the number of points in $X(k)$ is divisible by p. The trivial solution $(0, \ldots, 0)$ is always in $X(k)$, thus we must have at least $p - 1$ nontrivial solutions.

Because the polynomials are homogeneous, the solutions come in lines through the origin, so we can guarantee only one nontrivial solution up to scalar multiple. $\square$

The next example shows that the degree bound of Theorem 1.18 is sharp. It shows that over most fields, there are hypersurfaces in $\mathbb{P}^n$ of any degree $d > n$ admitting no rational points.

EXAMPLE 1.19. Fix a ground field k and let k' be a Galois extension of k, say of degree d. Let $\lambda_1, \ldots, \lambda_d$ be a k-basis of k'. Set

$$f(x_1, \ldots, x_d) := \prod_{\sigma\in \mathrm{Gal}\, k'/k} \left(\lambda_1^\sigma x_1 + \cdots + \lambda_d^\sigma x_d\right),$$

where the right hand side is the norm of the linear form $\lambda_1 x_1 + \cdots + \lambda_d x_d$, that is, the product of all its conjugates over k. Because f is invariant under the Galois group action of k'/k, it is defined over k, and so is the corresponding degree d hypersurface in $\mathbb{P}^{d-1}$. Geometrically, this hypersurface is the union of d hyperplanes defined over k', conjugate over k.

We claim that this hypersurface has no k-points. Indeed, a k-point would amount to a nontrivial k-solution of f, say $(p_1, \ldots, p_d)$. This, in turn, would mean a linear relation

$$\lambda_1^\sigma p_1 + \cdots + \lambda_d^\sigma p_d = 0 \quad \text{for some } \sigma \in \mathrm{Gal}(k'/k).$$

Because the set $\{\lambda_1, \ldots, \lambda_d\}$ is linearly independent over k, so is the conjugate set. This forces each p_i to be zero, and the solution was trivial after all. So the constructed degree d hypersurface in $\mathbb{P}^{d-1}$ has no k-points.

To create hypersurfaces of higher degree with no k-points, simply set some variables equal to zero to obtain examples of forms of degree d in fewer variables which admit no nontrivial solutions. In particular, the corresponding hypersurfaces can not be rational, or even unirational, over k.

The following exercise should convince the reader that the previous example can be used to generate examples over many familiar fields, including $\mathbb{Q}$ and $\mathbb{F}_p$.

EXERCISE 1.20. Let K be a field finitely generated over an algebraically closed subfield k. Prove that K has separable extensions of any degree. Prove that the same holds when the subfield k is a prime field, that is, when $k = \mathbb{Q}$ or $\mathbb{F}_p$.

Of course, we should not be fooled by Exercise 1.20 into thinking that every field has extensions of every degree. Obviously, algebraically closed fields, such as $\mathbb{C}$, have no nontrivial algebraic extensions. Likewise, the field of real numbers $\mathbb{R}$ has an extension of degree two, but no others.

Although Example 1.19 guarantees the existence of high degree hypersurfaces with no rational points, it is not completely satisfying because the hypersurfaces we constructed were not smooth. The next exercise, suggested by N. Katz, produces smooth examples, albeit not for every degree.

EXERCISE 1.21. Let p be a prime number and let X be the hypersurface in $\mathbb{P}^{p-2}$ defined by the homogeneous polynomial $x_1^{p-1} + \cdots + x_{p-1}^{p-1}$. Show that X is a smooth hypersurface of degree $p-1$ that does not have $\mathbb{F}_p$-points.

REMARK 1.22. It is not the case that there exist *smooth* hypersurfaces without $\mathbb{F}_q$-points of every degree d in $\mathbb{P}^n$ with $d > n$. Indeed, let $C \subset \mathbb{P}^2$ be a smooth curve of degree three and let k be a finite field of q elements. The Hasse–Weil estimates give that

$$\#C(k) \geq q + 1 - 2\sqrt{q} > 0.$$

(See, for instance, Hartshorne (1977, App. C) for a summary of the Weil conjectures.) In general, the Weil estimates show that a smooth hypersurface of degree d has a point in $\mathbb{F}_q$ whenever q is very large relative to d.

We now give a different proof of Theorem 1.17. As we pointed out immediately after its statement, the proof of Theorem 1.17 reduces to the case of a smooth conic in $\mathbb{P}^2$. Such a conic is isomorphic to the projective line over $\bar{k}$, so it suffices to prove the following theorem in the case where $n = 1$.

THEOREM 1.23. *Let X be a variety defined over a finite field k of cardinality q. Assume that X is isomorphic to $\mathbb{P}^n$ over $\bar{k}$. Then the set of k-points of X has cardinality*

$$1 + q + q^2 + \cdots + q^n.$$

Furthermore, the variety is in fact isomorphic to $\mathbb{P}^n$ over k.

A key idea in the proof is the use of the Frobenius morphism, developed in the next exercise.

EXERCISE 1.24. Let k be a finite field of cardinality q. Define the Frobenius morphism $F : \mathbb{P}^n \to \mathbb{P}^n$ by

$$(x_0 : \cdots : x_n) \mapsto \left(x_0^q : \cdots : x_n^q\right).$$

Prove the following properties:

1. F is one-to-one on closed points;
2. F has degree q^n;
3. the fixed points of F are exactly the points of $\mathbb{P}^n(k)$;
4. if $X \subset \mathbb{P}^n$ is a subvariety defined over k, then F restricts to a morphism $F : X \to X$;
5. the restriction of F to X does not depend on the choice of the embedding $X \subset \mathbb{P}^n$.

Exercise 1.24 suggests that we may be able to compute $X(k)$ as a fixed point set of the Frobenius morphism for any variety X. In turn, the cardinality of the fixed point set can be computed as an intersection number. Indeed, let $\Delta \subset X \times X$ denote the diagonal and let $\Gamma \subset X \times X$ denote the graph of the Frobenius morphism F on X. Then

$$X(k) \cong \Delta(k) = \Delta(\bar{k}) \cap \Gamma(\bar{k}).$$

So the cardinality of the set of k-points is equal to the intersection number $\Delta \cdot \Gamma$, assuming the intersection is transverse. The computation of this intersection number hinges on our ability to give good geometric descriptions of Δ and of Γ. This is our method for proving the next proposition, which establishes Theorem 1.23 in the curve case, and hence Theorems 1.17 and 1.16 in general.

Proposition 1.25. *A smooth plane conic over a finite field k of cardinality q admits exactly $q + 1$ points defined over k.*

Proof. Let C denote such a conic. According to the discussion above, the cardinality of $C(k)$ is equal to the intersection number $\Gamma \cdot \Delta$, assuming the intersection is transverse. This intersection number can be computed over $\bar{k}$. Since C is isomorphic to $\mathbb{P}^1$ over $\bar{k}$, the product $C \times C$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ over $\bar{k}$, and Δ, Γ are divisors on this smooth surface. Every divisor on $\mathbb{P}^1 \times \mathbb{P}^1$ is linearly equivalent to a divisor of the form $aE_1 + bE_2$, where $E_1 = p_1 \times \mathbb{P}^1$ and $E_2 = \mathbb{P}^1 \times p_2$, for any given points p_1, p_2 in $\mathbb{P}^1$.

We compute the linear equivalence class of both Δ and Γ over $\bar{k}$ by writing each in the form $aE_1 + bE_2$ and solving for a and b in each case. For Δ, note that $\Delta \cap E_1 = (p_1, p_1)$ and $\Delta \cap E_2 = (p_2, p_2)$, and that both intersections are transverse. This implies that $\Delta \sim E_1 + E_2$. For Γ, we have that $\Gamma \cap E_1$ and $\Gamma \cap E_2$ both consist of a single point, but transversality no longer holds. Choose suitable affine coordinates s and t for an affine chart $\mathbb{A}^2$ contained in $\mathbb{P}^1 \times \mathbb{P}^1$ so that E_1 and E_2 are given by the vanishing of s and t respectively, and Γ is given by the vanishing of $t - s^q$. Then we compute that $\Gamma \cdot E_1 = 1$ and $\Gamma \cdot E_2 = q$. This gives that $\Gamma \sim qE_1 + E_2$.

It is now straightforward to compute that

$$\Delta \cdot \Gamma = (E_1 + E_2) \cdot (qE_1 + E_2) = 1 + q.$$

Furthermore, it can be checked that the intersection of Δ and Γ is transverse, so it consists of precisely $1 + q$ distinct k-points. □

EXERCISE 1.26. The reader familiar with the notion of rational equivalence and the intersection theory of $\mathbb{P}^n \times \mathbb{P}^n$ should generalize the proof of Proposition 1.25 to higher dimension. This completes the proof of Theorem 1.23 in higher dimensions.

Finally, there are many ways to prove the final statement of Theorem 1.23 claiming that, in fact, X is isomorphic to $\mathbb{P}^n$ over k. The simplest is to build up $\mathbb{P}^n$ as follows.

Fix an isomorphism of X with $\mathbb{P}^n$ defined over $\bar{k}$. Pick two k-points P_0 and P_1 in X. The unique line L_1 through them in $\mathbb{P}^n \cong X$ is also defined over k. Next take a k-point P_2 not lying on L_1. Again, the unique plane L_2 spanned by the k-points P_0, P_1, P_2 is defined over k. Note that $L_1 \subset L_2$ and the linear system $\mathcal{O}_{L_2}(L_1)$ maps L_2 isomorphically to $\mathbb{P}^2$ over k. Now we continue, building up bigger and bigger linear spaces defined over k inside $X \cong \mathbb{P}^n$. The precise value of the cardinality of $X(k)$ is needed to prove that we can always choose k-points not in L_i for $i < n$. Eventually, we see that X is all of $\mathbb{P}^n$, as a variety over k.

A smooth projective variety that becomes isomorphic to projective space over the algebraic closure of the ground field is called a *Severi–Brauer variety*. A concise introduction to the theory of Severi–Brauer varieties, including a proof of the following result, can be found in Serre (1979, X.6).

THEOREM 1.27. *A Severi-Brauer variety is isomorphic to projective space if and only if it admits a rational point.*

1.5 Cubic hypersurfaces

After quadrics, the next natural examples are cubics. It turns out that the small change in degree leads to a much more interesting theory with many unsolved questions.

Cubic plane curves are usually called elliptic curves. The smooth ones are not rational. There are many ways to see this; for example, over $\mathbb{C}$, the elliptic curves have genus one, so can not be isomorphic to the projective line, which has genus zero. The theory of elliptic curves is very rich, and there are many nice books on the subject. See, for instance, Reid (1988, I.2) or Shafarevich (1994, III.3) for introductions.

Cubic surfaces were extensively studied in the nineteenth and early twentieth centuries, but their theory still has plenty of unsolved problems. In contrast to cubic curves, every smooth cubic surface over an algebraically closed field is rational. The point is that every such surface is isomorphic to the projective plane blown-up at six points. If the six points are defined over some algebraically non-closed subfield k, then surface is rational over k, but in general, rationality of cubic surfaces is a subtle question over algebraically non-closed fields.

In this section, we concentrate on examples of rational and unirational cubic surfaces over non-algebraically closed fields. Our goal is to find simple geometric conditions that imply rationality or unirationality for cubic hypersurfaces. One key fact we use is that every cubic surface contains exactly twenty-seven lines (defined over an algebraically closed field). This fact, as well as other basic facts about cubic surfaces, can be found in many introductory texts, including such as Reid (1988, §7) or Shafarevich (1994, IV.2.5).

We first note that rationality questions for singular cubics are relatively easy to answer.

Example 1.28. An irreducible cubic hypersurface in projective space (that is not a cone over a cubic hypersurface of lower dimension) is rational over k if it has a *singular* k-point.

Proof. Let X be the cubic hypersurface in $\mathbb{P}^{n+1}$, defined over the ground field k. Project from the singular point $P \in X(k)$ onto a general hyperplane defined over k. Since P has multiplicity two on X, any line through P has a unique third point of intersection with X. Its projection onto the hyperplane gives the one-to-one map from X to $\mathbb{P}^n$. Of course, this makes sense only when the line through P does not lie on X. This is where we use the assumption that X is not a cone: in this case, the generic line through P does not lie on X. □

It is also easy to find examples of cubics that are not unirational, as shown by the next exercise.

Exercise 1.29. Let $f(x, y)$ be a homogeneous cubic equation over $\mathbb{Z}$ that has no nontrivial roots modulo some prime p. (For instance, $x^3 - xy^2 + y^3$ works for $p = 2$ or 3.) Show that

$$f(x_0, x_1) + pf(x_2, x_3) = 0$$

has no rational solutions. Conclude that the corresponding cubic surface is not unirational over $\mathbb{Q}$.

It is harder to get examples of cubic surfaces which do have points but are still not rational. The simplest examples use topological considerations over the reals.

EXAMPLE 1.30. Let X be a cubic surface in real projective three space, defined by an equation (in affine coordinates) $x^2 + y^2 = f_3(z)$, where f_3 has three distinct real roots. Then X is not rational over $\mathbb{R}$.

PROOF. The set of points in $\mathbb{R}$ where f_3 takes positive values has two disjoint components. The equation $x^2 + y^2 = f_3(z)$ has real solutions (in fact, a circle's worth) if and only if $f_3 \geq 0$, so we see that, as a real manifold, $X(\mathbb{R})$ has two connected components. But if X is birationally equivalent to $\mathbb{P}^2$ over $\mathbb{R}$, then because $\mathbb{P}^2(\mathbb{R})$ is connected as a real manifold, so would be $X(\mathbb{R})$ (see Exercise 1.31 below.) $\square$

EXERCISE 1.31. Let X and Y be smooth projective varieties over $\mathbb{R}$. Show that if X and Y are birational then $X(\mathbb{R})$ and $Y(\mathbb{R})$ have the same number of connected components.

On the other hand, the cubic surface X of Example 1.30 is unirational over $\mathbb{R}$. We leave the reader the pleasure of finding a map $\mathbb{P}^2_{\mathbb{R}} \dashrightarrow X$ that is two-to-one onto one of the manifold components and misses the other component entirely. Indeed, the preimage of a point in the missed manifold component can be interpreted as a pair of complex conjugate points in the complex manifold $\mathbb{P}^2(\mathbb{C})$.

The following example, due to Swinnerton-Dyer, gives a fascinating specific example of a cubic surface which is not rational.

EXAMPLE 1.32 (Swinnerton-Dyer, 1962). Consider the cubic surface defined by

$$t(x^2 + y^2) = (4z - 7t)(z^2 - 2t^2).$$

1. The real points of this surface consist of two connected components.
2. On one manifold component, $\mathbb{Q}$-points are dense.
3. On the other manifold component, there are no $\mathbb{Q}$-points.

To see that the surface has two real components, consider the affine chart where $t = 1$. Setting $v^2 = x^2 + y^2$, we see that the surface is a surface of revolution for the elliptic curve

$$v^2 = (4z - 7)(z^2 - 2).$$

Because the function $f(z) = (4z - 7)(z^2 - 2)$ has three distinct real roots, we know from Example 1.30 that the curve, and hence the surface of revolution, has two disjoint real components. The two real components correspond to $z \geq 7/4$ and $|z| \leq \sqrt{2}$ respectively.

On the real component where $z \geq 7/4$, the $\mathbb{Q}$-points are dense. Indeed, this component contains $(x : y : z : t) = (1 : 1 : 2 : 1)$. The tangent plane to the surface at this point intersects with the surface to produce an irreducible singular cubic on S. This curve is rational over $\mathbb{Q}$, and its $\mathbb{Q}$-points are dense among its $\mathbb{R}$-points. In particular, the $\mathbb{Q}$-values for z are dense among all real values for $z \geq 7/4$. For each of these fixed $\mathbb{Q}$-values z_0, the plane $z = z_0$ intersects the surface S in the circle $x^2 + y^2 = f(z_0)$. This conic is $\mathbb{Q}$-rational and its $\mathbb{Q}$-points are dense among its $\mathbb{R}$-points. Thus the $\mathbb{Q}$-points of S are dense on the manifold component where $z/t \geq 7/4$.

There are no $\mathbb{Q}$-points on the component where $|z/t| \leq \sqrt{2}$. To see this, suppose that $(x : y : z : t)$ is such a $\mathbb{Q}$-point, where, without loss of generality, t and z are assumed relatively prime integers, with $t > 0$. So

$$t(7t - 4z)(2t^2 - z^2) = (tx)^2 + (ty)^2$$

is an integer which is the sum of two rational squares. Thus any prime p congruent to 3 modulo 4 that divides $t(7t - 4z)(2t^2 - z^2)$ must divide it an even number of times.

Because $|\frac{z}{t}| \leq \sqrt{2}$, each of the integer factors

$$t, \ (7t - 4z), \ (2t^2 - z^2)$$

is *positive*. We claim that none is congruent to 3 modulo 4. Indeed, no prime p congruent to 3 modulo 4 can divide any one of these factors to an odd power. For if some such p does, then it must divide precisely two of the factors an odd number of times. But because t and z are relatively prime, it follows that t and $2t^2 - z^2$ are relatively prime, and the only possible common prime factor of t and $(7t - 4z)$ is 2. Furthermore, if p divides both $(7t - 4z)$ and $(2t^2 - z^2)$, then p divides $(8t + 7z)(7t - 4z) - 28(2t^2 - z^2) = 17tz$. Since such p divides neither z nor t, the only possibility is $p = 17$, which is not congruent to 3 modulo 4.

Now if t is even, then z must be odd, but this would force $(2t^2 - z^2)$ to be congruent to 3 modulo 4. On the other hand, if t is odd, then it must be congruent to 1 modulo 4, but this forces $(7t - 4z)$ to be congruent to 3 modulo 4. This contradiction implies that there is no $\mathbb{Q}$-rational point on the component of the surface where $|z/t| \leq \sqrt{2}$.

We now investigate general geometric criteria for rationality or unirationality of smooth cubics.

EXAMPLE 1.33 (Rationality of cubics containing linear spaces). If a smooth cubic hypersurface of even dimension contains two disjoint linear spaces, each of half the dimension, then the cubic hypersurface is rational. In particular, a smooth cubic surface is rational over k if it contains two skew lines defined over k (of the twenty-seven lines on the surface defined over $\bar{k}$).

PROOF. Let $X \subset \mathbb{P}^{2n+1}$ be the cubic hypersurface, and let L_1 and L_2 be the two linear spaces on X. Consider the map

$$\begin{aligned} \phi : L_1 \times L_2 &\dashrightarrow X \\ (P, Q) &\mapsto \text{third intersection point } X \cap \overline{PQ}. \end{aligned}$$

This defines a birational map from $L_1 \times L_2$ to X. The map is well defined because a typical line intersects X in exactly three points (counting multiplicities). This map is birational: if the preimage of $x \in X$ includes two distinct pairs (P_1, Q_1) and (P_2, Q_2) on $L_1 \times L_2$, then the projections of the linear spaces L_1 and L_2 from x onto a general hyperplane would intersect each other in more than one point, which is impossible (see Example 1.36 for a more general discussion). Because

$$\mathbb{P}^{2n} \dashrightarrow \mathbb{P}^n \times \mathbb{P}^n \dashrightarrow L_1 \times L_2 \dashrightarrow X$$

are birational equivalences, we conclude that X is rational. Note that all maps above are defined over the ground field k. □

EXERCISE 1.34. 1. Find examples of smooth cubic hypersurfaces in $\mathbb{P}^{2n+1}$ containing two disjoint n-planes.
2. What is the dimension of the variety of all such cubics?
3. Why have we not considered linear spaces of nonequal dimension?
4. Write down a birational map between $\mathbb{P}^2$ and the cubic surface defined by the homogeneous polynomial $x^2y + y^2z + z^2v + v^2x$.

EXAMPLE 1.35 (Rationality of cubics containing conjugate linear spaces). We get an interesting variant of Example 1.33 for a cubic hypersurface of even dimension containing a pair of disjoint linear spaces each of half the dimension, which are defined over some quadratic extension $k' = k(\alpha)$ and conjugate to each other over k. Let $\bar{\alpha}$ denote the conjugate of α.

As in the previous example, we obtain a birational map $L_1 \times L_2 \dashrightarrow X$ defined over k'. How do we get down to k?

Here is the trick. Choose an affine chart $\mathbb{A}^{2n+1} \cong U \subset \mathbb{P}^{2n+1}$ intersecting L_1 and L_2. We can write the k'-points of $L_1 \cap U$ in the form

$$C \cdot \mathbf{w} + \mathbf{c},$$

where $\mathbf{w}$ is a k'-vector of $\mathbb{A}^n$, C is a $(2n+1) \times n$ matrix over k' and $\mathbf{c}$ is a k'-vector of $\mathbb{A}^{2n+1}$. Writing $\mathbf{w} = \mathbf{u} + \alpha\mathbf{v}$ where $\mathbf{u}$ and $\mathbf{v}$ are k-vectors of $\mathbb{A}^n$, and similarly for C and $\mathbf{c}$, we can rewrite this as

$$\Phi(\mathbf{u}, \mathbf{v}) := [A \cdot (\mathbf{u}, \mathbf{v}) + \mathbf{a}] + \alpha[B \cdot (\mathbf{u}, \mathbf{v}) + \mathbf{b}],$$

where A, B are $(2n+1) \times 2n$ matrices over k, and $\mathbf{a}$, $\mathbf{b}$ are k-vectors of $\mathbb{A}^{2n+1}$. This gives the representation

$$\bar{\Phi}(\mathbf{u}, \mathbf{v}) := [A \cdot (\mathbf{u}, \mathbf{v}) + \mathbf{a}] + \bar{\alpha}[B \cdot (\mathbf{u}, \mathbf{v}) + \mathbf{b}],$$

for k'-points in $L_2 \cap U$. Since $\Phi(\mathbf{u}, \mathbf{v})$ and $\bar{\Phi}(\mathbf{u}, \mathbf{v})$ are conjugate points over k, the line connecting them is defined over k. Now the intersection of this line with the cubic has a unique third intersection point which is necessarily defined over k. This gives a birational map $\mathbb{A}^n \times \mathbb{A}^n \dashrightarrow X$, defined over k.

For an explicit example consider the cubic surface in $\mathbb{P}^3$ defined by $x^3 + y^3 + z^3 = v^3$.

It is easy to see that this surface does not contain any disjoint pair of lines defined over $\mathbb{Q}$ (or even over $\mathbb{R}$) but that it does contain the conjugate pair of disjoint lines parameterized as $L_i = (w, -\epsilon_i w, \epsilon_i)$, where ϵ_i for $i = 1, 2$ are the complex cube roots of 1 and we work in the affine chart $v \neq 0$. Setting $w = t + \epsilon_1 s$ we obtain conjugate representations for the lines as

$$\Phi_i(s, t) = (t, s, 0) + \epsilon_i(s, s - t, 1).$$

The line joining them has a parametric representation with parameter λ:

$$(t, s, 0) + \lambda(s, s - t, 1).$$

Working out the third intersection point explicitly (a computation best done by computer) gives the birational map $\Phi : (s, t) \mapsto (x : y : z : 1)$ given by

$$\begin{aligned} x &= t + sz \\ y &= s + (s - t)z \\ z &= \frac{t^3 - 1 + s^3}{-2s^3 - 3st^2 + t^3 - 1 + 3s^2 t}. \end{aligned}$$

Example 1.36 (Unirationality of cubics). More generally, given any two subvarieties, U and V, of a cubic hypersurface X, one is tempted to form a similar map:

$$\begin{aligned} \phi : U \times V &\dashrightarrow X \\ (u, v) &\mapsto \text{third intersection point } X \cap \overline{uv}. \end{aligned}$$

If U and V are disjoint, this map is a morphism except at pairs of points (u, v) spanning a line on X; in general, it is not defined on $U \cap V$.

The map ϕ can not be dominant unless $\dim U + \dim V \geq \dim X$ and it can not be generically finite unless $\dim U + \dim V = \dim X$. When ϕ is finite, how does one compute its degree?

To determine the preimage of a general point $x \in X$, consider the projection π_x from $x \in X$ to a general hyperplane. The set $\pi_x(U) \cap \pi_x(V)$ consists of all points $\pi_x(u) = \pi_x(v)$, with u, v, and x collinear. In this case, assuming that $u \neq v$, the points $(u, v) \in U \times V$ are the preimages of x under ϕ. So, if $U \cap V = \emptyset$, we expect that the degree of ϕ is the cardinality of $\pi_x(U) \cap \pi_x(V)$. More generally, we must subtract something for the intersection points of U and V.

In Example 1.33, we applied this idea with U and V linear subspaces and deduced that cubic hypersurfaces are rational if they contain two disjoint linear subvarieties of half the dimension. More generally, the idea is useful for detecting *unirationality* of some cubics, as show by the proof of the following theorem of B. Segre.

THEOREM 1.37. *A smooth cubic surface over an infinite field k is unirational if and only if it admits a rational point over k.*

Recall that a point on a smooth cubic surface is called an Eckardt point if it is the intersection of three of the twenty-seven lines on the surface. Clearly, Eckardt points are quite special: a cubic surface can have at most finitely many Eckardt points and a general cubic surface contains none (Eckardt, 1876). We prove here the following special case Segre's theorem: *A smooth cubic surface over a perfect infinite field k is unirational if and only if it admits a k-rational point that is not an Eckardt point.* This case was proved by Segre (1943). To deduce the theorem in its full strength, one must apply the following later result of Segre (1951): if a cubic surface over an infinite field k contains a k-point, then it contains infinitely many k-points. Since any particular cubic surface has at most finitely many Eckardt points, this reduces the problem to our special case.

PROOF. First note that a smooth cubic surface in $\mathbb{P}^3$ containing two non-coplanar rational curves is unirational. Indeed, let C_1 and C_2 be rational curves on the surface X, and define the map $\phi : C_1 \times C_2 \dashrightarrow X$ as in Example 1.36. Because C_1 and C_2 do not lie in the same plane, their join (meaning the locus of points lying on lines joining points on C_1 to points on C_2) must be all of $\mathbb{P}^3$. This ensures that the map ϕ is dominant, and hence finite. Because C_1 and C_2 are rational (over k), we conclude that X is unirational (over k).

Thus we must find two non-coplanar rational curves on our cubic surface. Suppose first that the surface contains two k-rational points p_1 and p_2 that are not Eckardt points. Intersecting X with the tangent planes at each of the two k-points, we get two curves $C_1 = T_{p_1}X$ and $C_2 = T_{p_2}X$ on X defined over k. Typically, C_1 and C_2 are irreducible, hence each is rational over k by Example 1.28 (if C_1 is a cone over three points in $\mathbb{P}^1$, then p_1 is an Eckardt point). Furthermore, in this typical case, these plane cubics are not coplanar. If they were, then $T_{p_1}X$ and $T_{p_2}X$ would coincide, so that also $C_1 = C_2$ would be an irreducible plane cubic with two singular points p_1 and p_2, which is impossible. Even in the degenerate case where C_1 or C_2 is reducible, the argument often goes through unchanged. Indeed, if C_1 breaks up as a smooth conic together with a line (over k), then each of these components is rational over k, and by similar reasons, can not be coplanar with the irreducible curve C_2. A genuine exception can occur when $C_1 = C_2$ is a union of a line and a smooth plane conic over k, or when both C_1 and C_2 are a union of three lines (over $\bar{k}$).

However, these exceptional cases can be avoided as follows. Given one non-Eckardt k-point on a smooth cubic surface, the plane section C obtained by intersecting X with the tangent plane T_pX contains a rational component defined over k. Even in the case where C is a union of three lines meeting in distinct points p, p_1, p_2 over $\bar{k}$, we get a line defined over a k: since the Galois action of $\bar{k}/k$ must permute the three points but must fix p, it must also stabilize the line through p_1 and p_2, whence this line is defined over k.[1] The rational component of C gives us many rational points on X, and a general pair of them produces a pair of non-coplanar rational curves on X as in the previous paragraph. Thus we have shown that a smooth rational surface containing a rational point that is not an Eckardt point is unirational. □

It is only recently that Segre's result has been extended to finite fields and to all higher dimensional cubics.

THEOREM 1.38 (Kollár, 2002). *A smooth cubic hypersurface of dimension at least two is unirational over k if and only if it admits a k-point.*

Note that Theorem 1.38 is completely independent of the field: it holds over any ground field, infinite or not. We do not prove this theorem here, but instead refer to Kollár (2002).

The following two examples were explained to us by Swinnerton-Dyer.

EXERCISE 1.39. Check that the cubic surface defined by the equation

$$x_1^3 + x_1^2x_0 + x_1\left(x_0^2 + x_2^2 + x_3^2 + x_2x_3\right) + x_2^3 + x_2^2x_3 + x_3^3$$

[1] This is where we use the assumption that k is perfect; see Exercise 1.8 and the subsequent remark.

over the field $\mathbb{F}_2$ of two elements has exactly one $\mathbb{F}_2$ point. Show that up to a linear change of coordinates, this is the only example of a cubic surface over $\mathbb{F}_2$ which has precisely one point. (In fact, it turns out that this is the only such example over any finite field, see (Swinnerton-Dyer, 1981, 5.5).)

EXERCISE 1.40. Check that, up to a linear change of coordinates, the cubic surface defined by the equation

$$x_2\left(x_0^2 + x_0x_2 + x_2^2\right) + x_3\left(x_1^2 + x_1x_3 + x_3^2\right) + x_2^2x_3$$

is the only cubic surface over $\mathbb{F}_2$ which contains a line defined over $\mathbb{F}_2$ but no other $\mathbb{F}_2$-points.

EXAMPLE 1.41. An interesting variation on the map discussed in Example 1.36 is obtained by allowing $U = V$. For example, suppose that X is a smooth cubic four-fold in $\mathbb{P}^5$ containing a smooth surface S.

Consider the map

$$\phi : \frac{S \times S}{\sim} \dashrightarrow X$$
$$(P, Q) \mapsto \text{third point of intersection } X \cap \overline{PQ}.$$

Here, $\frac{S\times S}{\sim}$ is the symmetric product of S, the quotient variety of $S \times S$ by the action of the two-element group interchanging the factors. If S is unirational over k, then so is $S \times S$, and hence so is the image $\frac{S\times S}{\sim}$ under the generically two-to-one quotient map.

Consider a general point $x \in X$, say not on S. When is x in the image of ϕ? Consider the family of lines $\{\overline{sx}\}_{s\in S}$. The point x is in the image of ϕ precisely when at least one of these lines intersects S in a point other than s. In particular, the projection from x, $\pi_x : S \to S' \subset \mathbb{P}^4$ can not be one-to-one. Indeed, x has a unique preimage under ϕ precisely when the projection π_x collapses exactly two points of S to a single point. On the other hand, if $\pi_x : S \to S'$ is not of degree one, then x has infinitely many preimages under ϕ.

Thus ϕ is finite and dominant if and only if the generic projection of S from a point $x \in X$ is one-to-one except on a finite set. The next exercise provides one case where this condition can be verified.

EXERCISE 1.42. Let P_1, P_2, P_3, P_4 be four points in the projective plane, no three of them on a line. Let S be the surface obtained by blowing up these points. Show that the linear system of plane cubics through the four points gives an embedding $S \hookrightarrow \mathbb{P}^5$. (The image is called a degree five Del Pezzo surface. We will study these in Chapter 3.) Prove that a general projection of S to $\mathbb{P}^4$ has exactly one singular point. Use this to give some more examples of rational cubic four-folds.

Historical remark 1.43. The above method has an interesting history which illustrates the necessity of extreme care in counting dimensions.

One can see that the family of degree five Del Pezzo surfaces in $\mathbb{P}^5$ has dimension 35 and each is contained in a 24 dimensional family of cubics. Since the space of cubics in $\mathbb{P}^5$ has dimension 55, we might expect that every cubic in $\mathbb{P}^5$ contains a four-dimensional family of degree five Del Pezzo surfaces and is therefore rational. It turns out, however, that if a cubic in $\mathbb{P}^5$ contains a degree five Del Pezzo surface, it always contains a five-dimensional family of them, so the above argument is wrong.

A similarly incorrect "proof" of rationality of cubics in $\mathbb{P}^5$ using degree four ruled surfaces was published by Morin (1940).

It is still not known if the general cubic in $\mathbb{P}^5$ is rational or not. See Hassett (2000) for some recent related results.

1.6 Further examples of rational varieties

We begin with the rationality of determinantal varieties, which ultimately provides a different perspective on the fact that a cubic surface is geometrically rational.

Exercise 1.44. 1. Prove that the variety of $m \times n$ matrices of rank at most t is rational over any field. Find its nonsmooth locus.

2. Consider an $n \times n$ array of general linear forms on $\mathbb{P}^n$. Prove that the hypersurface defined by the determinant of this array is a rational variety. When is this variety smooth?

Theorem 1.45. *The equation of a smooth cubic surface over an algebraically closed field can always be written as the determinant of a* 3×3 *matrix whose entries are linear forms in four variables.*

The earliest proof of this fact appears in an 1866 paper of Clebsch, who credits Schröter (Clebsch, 1866). We give here a classical geometric proof. For a more algebraic proof using the Hilbert–Burch theorem, see Geramita (1989).

Proof. We use the configuration of the twenty-seven lines on the cubic surface S. We claim that there are nine lines on the surface that can be represented in two different ways as a union of three hyperplane sections. That is, there are six different linear functionals $l_1, l_2, l_3, m_1, m_2, m_3$ on $\mathbb{P}^3$ such that the hyperplane sections of S determined by each is a union of three distinct lines, and the nine lines obtained as hyperplane sections with the l_is are the same nine lines obtained from the m_is. Assuming this for a moment, the cubics $l_1l_2l_3$ and $m_1m_2m_3$ both define the same subscheme of S, which means that up to scalar, these cubics agree on S. In other words, the cubic

$$l_1l_2l_3 - \lambda m_1m_2m_3$$

is in the ideal generated by the cubic equation defining S, and hence it must generate it. On the other hand, the cubic $l_1l_2l_3 - \lambda m_1m_2m_3$ is the determinant of the matrix

$$\begin{pmatrix} l_1 & m_1 & 0 \\ 0 & l_2 & m_2 \\ -\lambda m_3 & 0 & l_3 \end{pmatrix}.$$

The proof will be complete upon establishing the existence of the special configuration of lines. First recall that S is the blowup of six points $P_1, P_2, \ldots, P_6$ in $\mathbb{P}^2$, no three on a line and no five on a conic. We embed S in $\mathbb{P}^3$ using the linear system of plane cubics through these six points. The twenty-seven lines on $S \subset \mathbb{P}^3$ are obtained as follows:

1. for each pair of two points P_i and P_j, the birational transform of the line $\overline{P_iP_j}$ in $\mathbb{P}^2$ joining them;
2. for each point P_i, the birational transform of the conic Q_i through the remaining five points;
3. for each point P_i, the fiber E_i over P_i.

For any pair of indices i, j, the three lines $\overline{P_iP_j}$, E_i, and Q_j form a (possibly degenerate) triangle on S. Indeed, thinking of the hyperplane sections of S as cubics in the plane through the six points, this triangle is the hyperplane section given by the cubic obtained as the union of $\overline{P_iP_j}$ and Q_j. Now it is easy to find such a configuration. For instance, the nine lines

$$\{E_1, Q_2, \overline{P_1P_2}\} \cup \{E_2, Q_3, \overline{P_2P_3}\} \cup \{E_3, Q_1, \overline{P_1P_3}\}$$

are the same as the nine lines

$$\{Q_1, E_2, \overline{P_1P_2}\} \cup \{Q_2, E_3, \overline{P_2P_3}\} \cup \{Q_3, E_1, \overline{P_1P_3}\},$$

with the groupings indicating the two different configurations of triangles. □

REMARK 1.46. Our proof of Theorem 1.45 uses the fact that a cubic surface is a blowup of $\mathbb{P}^2$ at six points, and therefore does not give a new proof that cubic surfaces are rational. However, the first proof that cubic surfaces are rational did proceed by showing first that they are determinantal. Indeed, the nineteenth century masters had such a detailed understanding of the configuration of lines on a cubic surface that they were able to see combinatorially that the special configuration of lines needed in the proof of Theorem 1.45 exists without using blowups.

The following exercises provide more examples of rational varieties.

EXERCISE 1.47. (1) Let $k = \mathbb{C}(t)$, and let X be a degree d hypersurface in $\mathbb{P}^n$ defined over k. Prove that if $d \leq n$, then X has at least one k-point. Find an example with exactly one k-point. For $d > n$, find a hypersurface with no k-points. Explain why such a hypersurface is nonrational.

(2) Now repeat this exercise, but with the ground field $k = \mathbb{C}(t, s)$ and $d^2 \leq n$.

EXERCISE 1.48. (1) Let $X \subset \mathbb{P}^1 \times \mathbb{P}^n$ be a smooth hypersurface of bidegree $(a, 2)$. When n is at least two, show that X is rational over $\mathbb{C}$.

(2) Let $X \subset \mathbb{P}^2 \times \mathbb{P}^n$ be a smooth hypersurface of bidegree $(a, 2)$. For $n \geq 4$, show that X is rational over $\mathbb{C}$.

EXERCISE 1.49. Let P be a point of multiplicity m on a hypersurface of degree d in $\mathbb{P}^n$. Show that there is an at least $(n + m - d - 2)$-dimensional family of lines contained in X and passing through P.

EXERCISE 1.50. Let X be a smooth hypersurface in $\mathbb{P}^n$ of degree $d \leq n$. Assuming the field is algebraically closed, find a rational curve passing through every point of X.

(It is an open question whether or not every such smooth hypersurface contains a rational surface through every point.)

EXERCISE 1.51. Let $X \subset \mathbb{P}^{3n+1}$ be a quartic hypersurface containing three linear spaces, each of dimension $2n$, whose common intersection is empty. Prove that X is rational.

1.7 Numerical criteria for nonrationality

Rationality and unirationality force strong numerical constraints on a variety. Let $\Omega_X = \Omega_{X/k}$ be the sheaf of regular differential forms (Kähler differentials) on a variety X over k.

THEOREM 1.52. *If a smooth projective variety X is rational, then it has no nontrivial global Kähler one-forms. In fact, the space of global sections $\Gamma(X, \Omega_X^{\otimes m})$ of the sheaf $\Omega_X^{\otimes m}$ is zero for all $m \geq 1$. The same holds for unirational X, provided the ground field has characteristic zero.*

Proof. Suppose we have a generically finite, dominant map $\phi : \mathbb{P}^n \dashrightarrow X$. Let $U \subset \mathbb{P}^n$ be an open set over which ϕ is defined; its complement may be assumed to have codimension at least two.

Nonzero differential forms on X pull back to nonzero differential forms on U, that is, we have an inclusion $\phi^*\Omega_X^{\otimes m} \hookrightarrow \Omega_U^{\otimes m}$. This is obvious when ϕ is birational, and easy to check when ϕ is finite (assuming k has characteristic zero). Because the complement of U has codimension at least two, the differential forms on U extend uniquely to forms on $\mathbb{P}^n$, inducing an identification

$$\Gamma\big(U, \Omega_U^{\otimes m}\big) = \Gamma\big(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^{\otimes m}\big).$$

Therefore $\Gamma(X, \Omega_X^{\otimes m}) \subset \Gamma(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^{\otimes m})$, and the problem is reduced to proving the vanishing for $\mathbb{P}^n$, left as an exercise. □

Exercise 1.53. Complete the proof by showing that $\Gamma(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^{\otimes m})$ is zero for an $m \geq 1$.

Remark 1.54. In prime characteristic, unirationality of X does not necessarily force the vanishing of the invariants $\Gamma(\Omega_X^{\otimes m})$. Indeed, the pull-back map for differential forms can be the zero map, so the argument above fails. For example, consider the Frobenius map F on $\mathbb{A}^n$ sending $(\lambda_1, \ldots, \lambda_n) \mapsto (\lambda_1^p, \ldots, \lambda_n^p)$, where $p > 0$ is the characteristic of the ground field. The induced map of differential forms $F^*\Omega \to \Omega$ sends every differential dx to $d(x^p) = px^{p-1}\,dx = 0$.

In characteristic p, we are led to the more sensible notion of *separable unirationality.* A variety X is separably unirational if there is a dominant generically étale map $\mathbb{P}^n \dashrightarrow X$; by *generically étale*, we mean that the map $\mathbb{P}^n \dashrightarrow X$ is generically finite and that the induced inclusion of function fields is *separable.* A generically finite morphism $f : Y \to X$ is separable if and only if the pull-back map on differentials $f^*\Omega_X^1 \to \Omega_Y^1$ is injective. This is the only property that we need in the proof of Theorem 1.52. With this in mind, the proof of Theorem 1.52 shows that $\Gamma(X, \Omega_X^{\otimes m}) = 0$ for a smooth projective separably unirational variety X of arbitrary characteristic.

For arbitrary X, the spaces $\Gamma(X, \Omega_X^{\otimes m})$ are usually hard to compute because the $\Omega_X^{\otimes m}$ have quite high rank. Therefore it is important to have similar criteria which involve line bundles only. The natural candidate is the canonical bundle $\omega_X = \wedge^n \Omega_X$ of highest degree Kähler differential forms, which is always defined over the fixed ground field. For smooth X, the canonical bundle is represented by a divisor K_X defined over the given ground field, and it is convenient to denote it by $\mathcal{O}_X(K_X)$.

Definition 1.55. The mth plurigenus of a smooth variety X is the dimension of the vector space of global section of the invertible sheaf $\mathcal{O}(mK_X)$. We denote this integer by $P_m(X)$.

The plurigenera are easily computable obstructions to rationality.

COROLLARY 1.56. *If X is a smooth projective variety that is separably unirational (for example, rational), then the plurigenera $P_m(X)$ vanish for all positive m.*

PROOF. Probably the easiest argument is to notice that the line bundles $\mathcal{O}(mK_X)$ also have a pull-back map $\phi^*\mathcal{O}_X(mK_X) \hookrightarrow \mathcal{O}_U(mK_U)$ as in the proof of Theorem 1.52. The rest of the proof applies verbatim.

Conceptually it is neater to use the injection $\mathcal{O}_X(K_X) \to \Omega_X^{\otimes \dim X}$ which comes from the vector space map $\wedge^{\dim V} V \hookrightarrow V^{\otimes \dim V}$ identifying the determinant with the corresponding multilinear form. $\square$

Looking at the proof of Corollary 1.56 we see that we have not used the full strength of our unirationality assumption. This leads to the following two weaker notions.

DEFINITION 1.57. A variety X is *ruled* if there exists a variety Y and a birational map $\phi : Y \times \mathbb{P}^1 \dashrightarrow X$. A variety X is *uniruled* if there exists a variety Y and a generically finite dominant map $\phi : Y \times \mathbb{P}^1 \dashrightarrow X$.

Loosely speaking, a variety is *uniruled* if it is covered by rational curves. Of course, every rational variety X is ruled, since $\mathbb{P}^n$ is birationally equivalent to $\mathbb{P}^{n-1} \times \mathbb{P}^1$.

As before, in positive characteristic there is another variant. The variety X is *separably uniruled* if there is a separable, generically finite map $Y \times \mathbb{P}^1 \dashrightarrow X$.

Every ruled variety is separably uniruled, and every unirational variety X is uniruled. In characteristic zero, separably uniruled is equivalent to uniruled.

THEOREM 1.58. *The plurigenera vanish for any smooth projective separably uniruled variety.*

PROOF. Let $\phi : Y \times \mathbb{P}^1 \dashrightarrow X$ be a separable uniruling of a smooth variety X. We need to show that $\Gamma(X, \Omega_X^{\otimes m}) = 0$ for all positive m. As before, we are reduced to proving that

$$\Gamma(Y \times \mathbb{P}^1, \mathcal{O}(mK_{Y\times\mathbb{P}^1})) = 0 \quad \text{for all } m \geq 1.$$

Notice that $K_{Y\times\mathbb{P}^1} = \pi_1^*K_Y + \pi_2^*K_{\mathbb{P}^1}$ where the π_i denote the coordinate projections. Thus

$$\Gamma(Y \times \mathbb{P}^1, \mathcal{O}(mK_{Y\times\mathbb{P}^1})) \cong \Gamma(Y, \mathcal{O}(mK_Y)) \otimes \Gamma(\mathbb{P}^1, \mathcal{O}(mK_{\mathbb{P}^1})).$$

Finally, $\Gamma(\mathbb{P}^1, \mathcal{O}(mK_{\mathbb{P}^1})) \cong \Gamma(\mathbb{P}^1, \mathcal{O}(-2m)) = 0$ and we are done. $\square$

EXERCISE 1.59. Show that the plurigenera of a smooth hypersurface of degree d in $\mathbb{P}^n$ do not vanish when $d > n$. Conclude that no smooth hypersurface whose degree exceeds its embedding dimension is separably uniruled. In particular, no such hypersurface is rational.

In positive characteristic, unirational but not separably unirational varieties exist, and in fact, there are unirational hypersurfaces of arbitrary degree.

EXERCISE 1.60. Show that a purely inseparable cover of a unirational variety over a perfect field is unirational.

REMARK 1.61. In characteristic p, the Fermat-type hypersurface defined by $x_0^n + x_1^n + \cdots + x_m^n$ in $\mathbb{P}^m$ is unirational if m is odd and some power of p is congruent to $-1 \mod n$. This is due to Shioda; the proof is elementary but tricky (Shioda, 1974). For further examples, see Shioda and Katsura (1979).

2

Cubic surfaces

In Chapter 1, we produced many examples of rational cubic surfaces. We also discussed the fact that a smooth cubic surface is unirational if and only if it has a rational point. In this chapter, we treat the subtle issue of rationality for smooth cubic surfaces more systematically. In particular, we show that there are many cubic surfaces defined over $\mathbb{Q}$ which are not rational over $\mathbb{Q}$.

We begin in Section one by stating our main results, which provide a complete understanding of rationality issues for smooth cubic surfaces of Picard number one. These results are Segre's theorem, stating that such a surface is never rational, and the related result of Manin stating that any two such birationally equivalent surfaces are actually isomorphic. In the second section, we set up the general machinery of linear systems to study birational maps of surfaces. This technique, called the *Noether–Fano method*, is quite powerful and ultimately leads to a proof that smooth quartic threefolds are not rational, in Chapter 5. In this chapter, however, we apply this method only to the case of cubic surfaces, proving the theorems of Segre and Manin in Section 3.

Over an algebraically closed field, every cubic surface has Picard number seven, and it is not obvious that there is any cubic surface with Picard number one. Thus, in Section 4, we develop criteria for checking whether a cubic surface has Picard number one. Using this, we show that a typical diagonal cubic surface has Picard number one over the rational numbers. Combined with Segre's theorem, we obtain many smooth cubic surfaces which are not rational over $\mathbb{Q}$.

In Section 5, we move away from cubic surfaces to take a closer look at birational self-maps of the projective plane. In particular, we prove a classical result of Noether and Castelnuovo stating that, over an algebraically closed field, every birational self-map of $\mathbb{P}^2$ factors as a composition of projective changes of coordinates and quadratic transformations.

2.1 The Segre–Manin theorem for cubic surfaces

Rationality for cubic surfaces is quite subtle. Our goal is to clarify the situation by proving the following theorem of B. Segre.

THEOREM 2.1 (Segre, 1942). *Let S be a smooth cubic surface over a field k. Assume that every curve $C \subset S$ is linearly equivalent to a hypersurface section. Then S is not rational (over k).*

It is not at all obvious that the assumptions of this theorem are ever satisfied. To clarify the situation, we consider the *Picard group* $\mathrm{Pic}(S)$ of S, that is, the group of divisors modulo linear equivalence.

The Picard group of a smooth cubic surface S over an algebraically closed field is isomorphic to $\mathbb{Z}^7$: thinking of S as the blowup of the projective plane at six points, the Picard group is freely generated by the six exceptional lines and the pull-back of the hyperplane class. An alternative derivation of this fact is given in Shafarevich (1994, IV.2.5). Related results are treated in Exercise 2.17.

On the other hand, the cubic surface S_k may be defined over some non-algebraically closed field k, even if the individual points we blow up are not defined over k. In this case, the Picard group of S_k may be smaller. Indeed, by Proposition 1.6, there is an injection

$$\mathrm{Pic}(S_k) \hookrightarrow \mathrm{Pic}(S_{\bar{k}}) \cong \mathbb{Z}^7,$$

and $\mathrm{Pic}(S_k)$ is frequently much smaller. A convenient measure of the size of $\mathrm{Pic}(S_k)$ is the *Picard number*, denoted by ρ_k. By definition, the Picard number of a smooth cubic surface over k is the rank of its Picard group.[1]

We can thus restate Theorem 2.1 as follows:

No smooth cubic surface of Picard number one is rational.

In Exercise 2.18, we outline how to construct examples of cubic surfaces over $\mathbb{Q}$ with Picard number one.

In Section 3, we prove the theorem of Segre. Essentially the same argument, with minor modifications to be made afterwards, proves the following stronger theorem of Manin (1966).

[1] For an arbitrary normal projective variety the Picard number is defined as the rank of the Néron–Severi group, the group of Cartier divisors up to numerical equivalence. For varieties with $h^1(X, \mathcal{O}_X) = 0$ the two definitions coincide.

THEOREM 2.2. *Two smooth cubic surfaces defined over a perfect field, each of Picard number one, are birationally equivalent if and only if they are projectively equivalent.*

CAUTION 2.3. Manin's theorem does not assert that every birational equivalence is a projective equivalence, and this is not at all true. It guarantees only that if two surfaces are birationally equivalent, then there exists an automorphism of $\mathbb{P}^3$ which maps one cubic surface into the other.

REMARK 2.4. The hypothesis of smoothness can not be weakened. For instance, consider a plane conic defined over k, together with six points on it conjugate, but not individually defined, over k. By blowing up the six points and then contracting the conic, we obtain a singular cubic surface with Picard number one. All such surfaces are birationally equivalent to each other, but two such are projectively equivalent if and only if the corresponding six-tuples of points are projectively equivalent in $\mathbb{P}^2$.

Similarly, the hypothesis that the Picard number is one is essential. For example, any two smooth cubic surfaces are birational over $\mathbb{C}$, but the isomorphism classes of smooth cubic surfaces form a four-dimensional family. Since cubic surfaces are embedded in $\mathbb{P}^3$ by the anticanonical linear system, every isomorphism between cubic surfaces is a projective equivalence.

2.2 Linear systems on surfaces

Let S be a smooth projective surface. A rational map

$$S \overset{\phi}{\dashrightarrow} \mathbb{P}^n$$

is given by an n-dimensional linear system of curves on S; in fact, such a map corresponds to a unique *mobile* linear system, where mobile means that the system has no fixed curves. Recalling that the *base locus* of a linear system is the intersection of all its members, we see that the base locus of a mobile linear system on a surface is simply the finite sets of points where the corresponding rational map is not defined.

Consider a birational morphism $f : S' \to S$. The *birational transform* of the mobile linear system Γ on S under f is the mobile linear system Γ' on S' whose general member is the birational transform of the general member of Γ. Equivalently, Γ' is the linear system obtained by pulling back Γ and then throwing away any fixed curves. In particular, to the extent to which S and S' are "the same," Γ and Γ' determine "the same" map to $\mathbb{P}^n$:

$$\begin{tikzcd} S' \arrow[d, "\pi"'] \arrow[dr, dashed, "\phi_{\Gamma'}"] & \\ S \arrow[r, dashed, "\phi_{\Gamma}"] & \mathbb{P}^n \end{tikzcd}$$

An important theme in this chapter, as well as in Chapter 5 where these methods are generalized to higher dimension, is the relationship of the linear systems Γ and K_S on S to the linear systems Γ' and $K_{S'}$ on S'. For surfaces, this relationship is expressed in terms of intersection numbers. The self-intersection number Γ^2 and the canonical intersection number $\Gamma \cdot K_S$ are of particular relevance.

2.5. Self-intersection numbers and birational maps. Let Γ be a mobile linear system on a smooth surface S. The self-intersection number of Γ–by which we mean the intersection number of two general members of Γ–is an important invariant of the map ϕ given by Γ.

When Γ is base point free, the map ϕ is a morphism and the self-intersection number of Γ is simply the degree of the image of ϕ times the degree of ϕ:

$$\Gamma^2 = \deg(\phi) \cdot \deg(\phi(S)). \tag{2.5.1}$$

In particular, when ϕ is a morphism onto $\mathbb{P}^2$, the self-intersection number Γ^2 is equal to the degree of ϕ.

When Γ has base points, the self-intersection number takes into account their multiplicities. The *multiplicity* of a linear system at a point is simply the multiplicity of a general member there. Thus the expected contribution to Γ^2 of a base point P of multiplicity m is m^2. However, this is valid only when two general curves in Γ have distinct tangents at P. The number is even higher if the curves share tangents at P; this is the case where Γ has base points *infinitely near* to P.

Consider the blowup $\pi : S' \to S$ of S at a base point P of Γ. The birational transform C' of a general member C of Γ intersects the exceptional fiber E in points corresponding to the tangent directions to the curve C at P. The base points of the birational transform Γ' of Γ that lie in E are called *base points of* Γ *infinitely near to* P. They represent the tangent directions at P that are shared by all members of Γ. (It is possible that Γ' has base points lying in E. Blowing them up, the base points of the birational transform of Γ' lying in the new exceptional fiber are also base points of Γ infinitely near to P. These correspond to higher order shared tangent directions of the members of Γ.)

Let us consider what happens to the self-intersection and canonical intersection numbers of Γ after blowing up one of its base points. For a general member C of Γ, it is easy to check that $\pi^*C = C' + mE$ where E is the exceptional fiber of the blowup and m is the multiplicity of Γ at P. Thus

$$\Gamma' = \pi^*\Gamma - mE \quad \text{and} \quad K_{S'} = \pi^*K_S + E,$$

where K_S and $K_{S'}$ denote the canonical classes of S and S' respectively. This leads to the following numerical relationship between intersection numbers on S and S':

EXERCISE 2.6. With notation as above, verify that

$$\Gamma'^2 = \Gamma^2 - m^2 \quad \text{and} \quad \Gamma' \cdot K_{S'} = \Gamma \cdot K_S + m.$$

2.7. RESOLUTION OF INDETERMINACY. Because the self-intersection number of the birationally transformed linear system drops with each blowup, the process of blowing up base points can be iterated until we have gotten rid of all base points. In this way, we arrive at a smooth surface $\bar{S}$, and a base point free linear system $\bar{\Gamma}$ defining the "same map" (i.e. after composition with the blowing up maps) to projective space as Γ.

$$\begin{tikzcd}
\bar{S} \arrow[d] \arrow[ddr, "\phi_{\bar{\Gamma}}"] & \\
\vdots \arrow[d] & \\
S \arrow[r, dashed, "\phi_\Gamma"] & \mathbb{P}^n
\end{tikzcd}$$

This process is called *resolving the indeterminacy* of the rational map ϕ_Γ. In this commutative diagram, each morphism in the tower on the left is a blowup at a base point.

Now suppose that Γ defines a *birational* map ϕ_Γ, and let T denote the image of S under this map. Resolving the indeterminacy of ϕ_Γ, we arrive at a birational *morphism*

$$\phi_{\bar{\Gamma}} : \bar{S} \to T \subset \mathbb{P}^n$$

onto the surface T. From formula (2.5.1), we know that the self-intersection number of $\bar{\Gamma}$ is equal to the degree of T. Therefore, after iterating the computation of Exercise 2.6, we arrive at the following formulas for intersection numbers on S.

EXERCISE 2.8. Verify that

$$\Gamma^2 - \sum m_i^2 = \deg T \quad \text{and} \quad K_S \cdot \Gamma + \sum m_i = H \cdot K_T$$

where the m_i are the multiplicities of all base points of Γ, including the infinitely near ones, and H is the restriction to T of the hyperplane class in $\mathbb{P}^n$.

Having carried out these computations, the following theorem is easy to prove.

THEOREM 2.9. *Let S be a smooth projective surface over k. Then S is rational over k if and only if S admits a mobile two-dimensional linear system Γ defined over k satisfying*

$$\Gamma^2 - \sum m_i^2 = 1 \quad \textit{and} \quad K_S \cdot \Gamma + \sum m_i = -3,$$

where the m_i are the multiplicities of all base points of Γ over $\bar{k}$, including the infinitely near ones.

We see from the proof that the second condition $K_S \cdot \Gamma + \sum m_i = -3$ can in fact be omitted. In our applications, the two numerical conditions together turn out to be very useful.

PROOF. Assume that S is rational over k, and let $\phi : S \dashrightarrow \mathbb{P}^2$ be a birational map. Let Γ be the mobile linear system on S obtained as the birational transform of the hyperplane system on $\mathbb{P}^2$. The dimension of Γ is two and the desired numerical conditions follow immediately from the formulas of Exercise 2.8.

Conversely, given a linear system Γ satisfying the given numerical conditions, it determines a rational map $\phi_\Gamma : S \dashrightarrow \mathbb{P}^2$ defined over k. We need only verify that this map is actually birational. Because the map is *a priori* defined over k, to check that it is birational we can assume that k is algebraically closed since whether or not the map is dominant and degree one is unaffected by replacing k by its algebraic closure.

Blow up the base points of Γ, including the infinitely near ones, to obtain a morphism $\bar{\phi} : \bar{S} \to \mathbb{P}^2$ resolving the indeterminacy of ϕ. The corresponding mobile linear system $\bar{\Gamma}$ has dimension two, and the numerical conditions $\bar{\Gamma}^2 = 1$ and $\bar{\Gamma} \cdot K_{\bar{S}} = -3$ hold. Because S and $\bar{S}$ are birationally equivalent, it is sufficient to show that the morphism $\phi_{\bar{\Gamma}} : \bar{S} \to \mathbb{P}^2$ is a birational equivalence.

To check that the map $\phi_{\bar{\Gamma}} : \bar{S} \to \mathbb{P}^2$ is surjective, assume, on the contrary, that its image is a plane curve, B. Then every member of $\bar{\Gamma}$ is a union of fibers of $\bar{S} \to B$. This forces the self-intersection number $\bar{\Gamma}^2$ to be zero, contradicting $\bar{\Gamma}^2 = 1$. Hence $\phi_{\bar{\Gamma}}$ is surjective.

Finally, since $\bar{\Gamma}^2 = 1$, the formula (2.5.1) implies that $\deg \phi_{\bar{\Gamma}} = 1$. Thus $\phi_{\bar{\Gamma}}$ is birational, and our surface S is rational. □

CAUTION 2.10. It is possible for a linear system on a variety to be defined over k even though its base points are not. For example, let $F(X) \in \mathbb{Q}[X]$. As

λ and μ vary through $\mathbb{C}$, the linear system of divisors given by the vanishing of the polynomials $\lambda Y + \mu F(X)$ is a one dimensional linear system on $\mathbb{A}^2$ defined over $\mathbb{Q}$. The zeros of $F(X)$ determine the base points, since (x, y) is a base point if and only if $(x, y) = (\alpha, 0)$ where α is a root of F. These base points may not be defined over $\mathbb{Q}$, although the linear system is defined over $\mathbb{Q}$. The map to projective space determined by this linear system is defined over $\mathbb{Q}$ even when its base points are not.

CAUTION 2.11. Given a mobile linear system on a smooth surface, it is not always possible to compute the degree of the image variety from the numerical data of Exercise 2.8 as we did in Theorem 2.9. For instance, if Γ is a three-dimensional linear system satisfying

$$\Gamma^2 - \sum m_i^2 = 4 \qquad \text{and} \qquad K_S \cdot \Gamma + \sum m_i = 0,$$

then there are two possibilities for Γ. Either it is a birational map whose image is a degree four surface in $\mathbb{P}^4$ or it is a degree two map to a quadric surface in $\mathbb{P}^3$.

We next record an important corollary of Theorem 2.9. Although it looks simple, it is the starting point for our later construction of families of higher dimensional nonrational Fano varieties in Chapter 5.

COROLLARY 2.12. *Let S be a smooth projective surface whose Picard group is generated by the canonical class K_S. Assume also that $K_S^2 = 1$. Then S is not rational.*

Theorem 3.36 classifies surfaces whose anticanonical class is ample and satisfies $K^2 = 1$ (called *degree one Del Pezzo* surfaces). Over a algebraically non-closed field, many such surfaces have Picard number one, so Corollary 2.12 is not vacuous. See Chapter 5, Section 3 for a higher dimensional analog of Corollary 2.12.

PROOF OF COROLLARY 2.12. Assume the contrary. Then we obtain a linear system $\Gamma \subset |-dK_S|$ satisfying the numerical conditions of Theorem 2.9. In particular,

$$d^2 = \Gamma^2 = 1 + \sum m_i^2 \quad \text{and} \quad d = -\Gamma \cdot K_S = 3 + \sum m_i.$$

The second equation says that $m_i < d$ for every i, hence we get a contradiction by

$$d^2 = 1 + \sum m_i^2 \le 1 + d \sum m_i = d^2 - 3d + 1 < d^2.$$

$\square$

2.3 The proofs of the theorems of Segre and Manin

The proof of the Segre–Manin results begins with the general observation that the Picard group of a smooth cubic surface of Picard number one is generated by the class of a hyperplane section. Indeed, the Picard group of S is torsion-free, because $\mathrm{Pic}(S) \subset \mathrm{Pic}\,(S_{\bar{k}}) \cong \mathbb{Z}^{\oplus 7}$. Also, the hyperplane class H is not divisible: if $H = mD$ for some divisor D and some integer m, then because $H^2 = 3 = m^2 D^2$, it follows that $m = 1$. Of course, for a cubic surface, the canonical class K_S is given by $-H$, so we can also say that the Picard group of a smooth cubic surface of Picard number one is generated by the canonical class.

Segre's theorem asserts that no cubic surface with Picard number one can be rational. If this were false, there would be a birational map $\phi : S \dashrightarrow \mathbb{P}^2_k$ defined over k, given by some mobile linear system Γ. Because the Picard group is generated by the hyperplane class H, the linear system Γ must be contained in the complete linear system $|dH|$ for some d. Therefore, the proof of Segre's theorem (Theorem 2.1) will be complete upon proving the following theorem.

THEOREM 2.13. *If $S \subset \mathbb{P}^3_k$ is a smooth cubic surface, then there is no mobile linear system on S contained in $|dH|$ that defines a birational map to the projective plane.*

Although the statement of this theorem is less appealing than Segre's theorem, we have, in effect, reduced the proof of Segre's theorem to the case where the ground field is algebraically closed: if such a linear system is defined over k, then it is also defined over the algebraic closure of k. Note that a naive reduction of Theorem 2.1 to the algebraically closed field case is not possible, as the Picard number is never one over an algebraically closed field.

PROOF OF THEOREM 2.13. Suppose that such a linear system, Γ, exists and defines the birational map $\phi_\Gamma : S \dashrightarrow \mathbb{P}^2_k$. Without loss of generality, we assume k is algebraically closed, as explained above.

Let $P_1, \ldots, P_r$ be the base points of Γ, including the infinitely near ones, and let $m_1, \ldots, m_r$ be their multiplicities. We first claim that some base point must have multiplicity greater than d. Indeed, from Theorem 2.9 and the fact that $K_S = -H$, we have

$$\begin{aligned} \sum m_i^2 &= \Gamma^2 - 1 = 3d^2 - 1 \quad \text{and} \\ \sum m_i &= -K_S \cdot \Gamma - 3 = 3d - 3. \end{aligned} \tag{2.13.1}$$

If all m_i are less than or equal to d, then

$$3d^2 - 1 = \sum m_i^2 \le d \sum m_i = 3d^2 - 3d < 3d^2 - 1.$$

This contradiction ensures that at least one base point has multiplicity greater than d.

Let P be a base point of Γ of multiplicity greater than d. There is no loss of generality in assuming that $P \in S$, that is, that P is an actual base point, not an infinitely near one. This is because the multiplicity of a base point is greater than or equal to the multiplicity of any base point infinitely near it. (In fact, the multiplicity of $P \in S$ as a base point of the linear system Γ is greater than or equal to the sum of the multiplicities of all base points of Γ' which lie over P, where Γ' is the birational transform of the linear system Γ under the blowing up map of S at P.)

Furthermore, the high multiplicity base point P can not lie on any line on S. Indeed, since $\Gamma \subset |dH|$, we must have that $L \cdot C \leq d$ for all lines L on S and all $C \in \Gamma$. Computing $C \cdot L$ as the sum over all points (with multiplicities) in $C \cap L$, we see that C can not have a multiple point of order more than d on L.

The proof of Theorem 2.13 proceeds by induction on d. The inductive step is accomplished by finding a birational self-map of S—in fact, a birational involution—that takes Γ to a linear system contained in the linear system $|d'H|$ with $d' < d$. We now construct this involution.

2.14. An involution of the cubic surface. First recall the following involution of a plane cubic curve E: fixing a point P on E, define the map τ which sends $Q \in E$ to the third point of intersection of E with $\overline{PQ}$. The map τ extends to an involution defined everywhere on E by sending the point P to the intersection of E with the tangent line through P.

We attempt to construct a similar involution of the cubic surface S in $\mathbb{P}^3$. Define a self-map τ of S as follows: fix a point P on S and for each Q in S, let $\tau(Q)$ be the third point of intersection of S with the line $\overline{PQ}$. This defines a rational map $\tau : S \dashrightarrow S$ such that $\tau^2 = id$. If we assume that S contains no lines through P, then τ is defined everywhere on S, except at P. However, unlike the situation of the plane cubic, there is a whole plane of tangent lines to S at P, so there is no way to extend τ to a morphism at P. Indeed, τ contracts the entire curve $D = T_P S \cap S$ to the point P on S.

As usual, the best way to sort out different tangent directions at a point is to blow up. Let $\pi : S' \to S$ be the blowup of S at P. Obviously, the involution τ extends to an involution $\tilde{\tau}$ of S': each point in the exceptional fiber corresponds to a tangent direction through the point P on S, and so there is a unique third point of intersection of the corresponding line with S.

To understand $\tilde{\tau}$ better, consider the following construction. By definition, the blowup of $\mathbb{P}^3$ at P consists of those points (x, ℓ) in $\mathbb{P}^3 \times \mathbb{P}^2$, where

$\mathbb{P}^2 = \mathbb{P}(T_P\mathbb{P}^3)$, such that $x \in \ell$. The blowup of S at P is identified with the corresponding birational transform of S. The blowing up map π is the projection of S' onto the first factor $S \subset \mathbb{P}^3$. Let q denote the projection of S' onto the second factor $\mathbb{P}^2$:

$$\begin{tikzcd} S' \arrow[r, hook] \arrow[d, "\pi"'] \arrow[dr, "q"] & S \times \mathbb{P}^2 \\ S \arrow[r, dashed] & \mathbb{P}^2 \end{tikzcd}$$

EXERCISE 2.15. Assume that P does not lie on any line on S. Show that q has degree two and ramifies along a smooth curve of degree four. Find the equation of this branch locus.

Now the following facts about $\tilde{\tau}$ are easily verified:

1. $\tilde{\tau}$ is the unique nontrivial (Galois) automorphism of the degree two cover S' of $\mathbb{P}^2$.
2. $\tilde{\tau}$ interchanges the exceptional divisor E of the blowup and the proper transform D' of the curve $D = T_P S \cap S$ contacted by τ to P.
3. $|\pi^* H - E| = |q^* L|$, where L is a line in $\mathbb{P}^2$ and H is a hyperplane section of S.

We are ready to complete the proof of Theorem 2.13. Begin with a mobile linear system Γ defining a map to the plane, and assume that Γ is contained in $|dH|$, with d as small as possible. As we have seen, Γ can be assumed to have a base point P on S of multiplicity m strictly larger than d.

Let $\Gamma' = \pi^*\Gamma - mE$ be the birational transform of Γ under the blowup $\pi : S' \to S$ at the base point P. Because $\Gamma \subset |dH|$, we have

$$\begin{aligned} \Gamma' + (m-d)E = \pi^*\Gamma - dE \subset |\pi^*(dH) - dE| &= |d(\pi^* H - E)| \\ &= |q^*(dL)|. \end{aligned}$$

Applying the automorphism $\tilde{\tau}$ to S', the elements of $\Gamma' + (m-d)E$ are taken to another linear system inside $|q^*(dL)|$ because $\tilde{\tau} \in \mathrm{Aut}(S'/\mathbb{P}^2)$ preserves any linear system pulled back from $\mathbb{P}^2$. Therefore, the linear system $\tilde{\tau}(\Gamma' + (m-d)E) = \tilde{\tau}(\Gamma') + (m-d)D'$ is contained in

$$|q^*(dL)| = |d(\pi^* H - E)| \subset |\pi^*(dH)|.$$

Pushing back down to S, we have

$$\tau(\Gamma) + (m - d)D \subset |dH|.$$

Because D is a hyperplane section of S, we conclude that

$$\tau(\Gamma) \subset |(d - (m - d))H|.$$

Finally, because $m > d$, the linear system $\tau(\Gamma)$ is contained in $|d'H|$, with $d' < d$.

Because the linear system $\tau(\Gamma)$ also defines a birational map to the projective plane, we have violated our minimality assumption on d. This contradiction completes the proof of Theorem 2.13 and hence the proof of Segre's theorem as well. $\square$

The arguments of Segre's theorem are easily altered to produce the following proof of Manin's theorem.

PROOF OF THEOREM 2.2. Assume that $\phi : S \dashrightarrow S'$ is a birational equivalence between cubic surfaces of Picard number one. Let Γ be the linear system on S obtained by pulling back the hyperplane system on S'. Thus Γ is the linear system of dimension three corresponding to the map ϕ.

Because the Picard number of S is one, we can assume, as before, that $\Gamma \subset |dH|$ for some d, where H is a hyperplane section of S. Let $P_1, \ldots, P_r$ be the base points of Γ, including the infinitely near ones, and suppose their multiplicities are $m_1, \ldots, m_r$. Again, we claim that some base point must have multiplicity greater than d. Indeed, because $H^2 = 3$ and $K_{S'} \cdot H = -3$, we again compute using Exercise 2.8:

$$\sum m_i = 3d - 3 \quad \text{and} \quad \sum m_i^2 = 3d^2 - 3.$$

If all $m_i \le d$, then $3d^2 - 3 = \sum m_i^2 \le d(\sum m_i) = 3d^2 - 3d$. This is possible only if $d = 1$, in which case ϕ is induced by an automorphism of $\mathbb{P}^3$, and the proof is complete.

Therefore, we may assume without loss of generality that some $m_i > d$, and the corresponding P_i may be assumed to be an honest point (as opposed to an infinitely near point) on S. The same trick that was used to accomplish the inductive step in the proof of Segre's theorem works here too. The only problem is that the involution τ is not defined over k unless the base point P is defined over k. Thus the previous argument shows only that if S and S' are birationally equivalent over k, then they are projectively equivalent over $\bar{k}$. This is a nontrivial result but not quite as strong as Manin's theorem.

To see that S and S' are actually projectively equivalent over k, we need to construct an involution τ defined over k. Because the Galois group of $\bar{k}$

over k acts on the P_i preserving multiplicities m_i, it follows from the equation $\sum m_i = 3d - 3$ that at most two of the base points P_i can have multiplicity greater than d. If exactly one, say P_1, has multiplicity greater than d, then the Galois group fixes this base point. Because k is perfect, this implies that P_1 is defined over k, so the involution τ is defined over k and the inductive step can be carried out as before. If exactly two base points, say P_1 and P_2, have multiplicity exceeding d, then the Galois group must fix their union, and so $P_1 \cup P_2$ is defined over k. As before, P_1 may be assumed to be on S. If P_2 is infinitely near P_1 then P_1 is defined over k. Thus we are left with the case when P_1 and P_2 are on S and they are conjugate over k. As before, neither P_1 nor P_2 lies on any line on the cubic surface and there is no conic containing both.

Consider the linear system $\Theta \subset |2H|$ of quadric sections on the cubic surface S passing through both P_1 and P_2 with multiplicity at least two. This linear system contains the symmetric square of the pencil of hyperplanes through P_1 and P_2 as well as the divisor $T_{P_1}S \cap S + T_{P_2}S \cap S$. These divisors generate all of Θ, which therefore has dimension three.

The only base points of Θ are P_1 and P_2. Indeed, since the line $\overline{P_1P_2}$ does not lie on S, the linear system of planes through P_1 and P_2 has a unique third base point P_0 where $\overline{P_1P_2}$ intersects S. But since P_0 can not lie in $T_{P_1}S \cap S + T_{P_2}S \cap S$, we see that Θ has exactly two base points, P_1 and P_2.

Let s_1 and s_2 be defining equations for the linear system of hyperplanes through $\overline{P_1P_2}$. Then defining equations for the generators of Θ are

$$u_0, \quad u_1 = s_1^2, \quad u_2 = s_1 s_2, \quad u_3 = s_2^2,$$

where u_0 is a defining equation for the divisor $T_{P_1}S + T_{P_2}S$. Thus the image of the rational map defined by Θ lies in the singular quadric surface Q given by the equation $u_1u_3 = u_2^2$.

Now consider the map

$$\phi_\Theta : S \dashrightarrow Q$$

given by Θ. We claim that this is a two-to-one map defined over k. Indeed, let H be a general plane through P_1 and P_2 and consider the plane section $E = S \cap H$. Then $\Theta|_E$ is the linear system $|2P_0|$, hence Θ gives a two-to-one map from the elliptic curve E to a ruling on the cone Q. We can use this map to define a birational involution of S interchanging the fibers of ϕ_Θ.

Finally, the proof of Manin's theorem is completed by arguing similarly as in the proof of Theorem 2.13: choose d smallest possible and then apply the involution constructed above to produce a smaller d. This contradiction proves the theorem. □

2.4 Computing the Picard number of cubic surfaces

In order to use Theorem 2.1 to produce explicit examples of nonrational cubic surfaces, we need criteria for detecting when a cubic surface has Picard number one. For higher degree surfaces, it is not easy to tell when the Picard number is one.[2] By contrast, the following theorem of Segre asserts that the Picard number of a cubic surface can be computed from geometric considerations about the twenty-seven lines.

THEOREM 2.16 (Segre, 1951). *Let S_k be a smooth cubic surface in $\mathbb{P}^3$ and consider the action of the Galois group of $\bar{k}/k$ on the twenty-seven lines of $S_{\bar{k}}$. The following are equivalent.*

1. *The Picard number $\rho_k(S)$ is one.*
2. *The sum of the lines in each Galois orbit is linearly equivalent to a multiple of the hyperplane class on S.*
3. *No Galois orbit consists of disjoint lines on S.*

PROOF. Assume for simplicity that k is perfect; the case where k is not perfect is relegated to Exercise 2.19.

Let K/k be a finite Galois extension such that all lines are defined over K. Set $G = \mathrm{Gal}(K/k)$.

Let L be any line on S_K. The sum over all conjugates $\sum_{\sigma \in G} \sigma(L)$ is invariant under G by construction, hence it is a divisor defined over k. (A weak version of this was proved in Exercise 1.8; the optimal version is proved later in Section 3.3.)

If the Picard number of S_k is one then, because the hyperplane class generates $\mathrm{Pic}(S_k)$, these orbit sums are all multiples of the hyperplane class $-K_S$. This establishes that (1) implies (2).

Next suppose that $\{L_1, \dots, L_t\}$ is an orbit consisting of non-intersecting lines. If (2) holds then $L_1 + \cdots + L_t \sim c \cdot K_S$ for some $c \in \mathbb{Q}$, hence

$$-t = (L_1 + \cdots + L_t)^2 = c^2 \cdot K_S^2 = 3c^2 \geq 0.$$

This contradiction proves that (2) implies (3).

Finally we prove that (3) implies (1). Let $D \subset S$ be an irreducible curve. First we prove that if L is any line on S then

$$L \cdot D = \tfrac{1}{3}(-K_S \cdot D). \tag{2.16.4}$$

[2] For every positive $d \geq 1$ there are degree four complex surfaces in $\mathbb{P}^3$ whose Picard group is generated by the hyperplane section H and a smooth rational curve C of degree d. It is not hard to see in this case that every effective divisor of degree less than d is a hypersurface section.

To see this consider all the lines $L_{i_1}, \dots, L_{i_t}$ such that

$$L_j \cdot D - \tfrac{1}{3}(-K_S \cdot D)$$

has maximal absolute value and the same sign for $i_1, \dots, i_t$. The set of these lines is invariant under G, hence by assumption (3) there are two lines, say L_{i_1}, L_{i_2} which intersect. Let $H \subset \mathbb{P}^3$ be the plane spanned by these two lines. The intersection $S \cap H$ consists of 3 lines L_{i_1}, L_{i_2}, L^*. Thus

$$-K_S \cdot D = H \cdot D = L_{i_1} \cdot D + L_{i_2} \cdot D + L^* \cdot D.$$

This can be rearranged to

$$L^* \cdot D - \tfrac{1}{3}(-K_S \cdot D) = -2 \left[L_{i_1} \cdot D - \tfrac{1}{3}(-K_S \cdot D)\right].$$

This contradicts the maximality of our choice, unless (2.16.4) holds. Since $L \cdot K_S = -1$ for any line L, the latter can be rewritten as

$$L \cdot D = L \cdot (-mK_S) \quad \text{where} \quad m = \tfrac{1}{3}(-K_S \cdot D).$$

By Exercise 2.17.2 this implies that D is linearly equivalent to $-mK_S$. This shows that the Picard number of S is one. □

We have already mentioned that the lines generate the Picard group of a cubic surface. We state below three versions of this result. The first two are easy to prove and they are all we used. The strongest statement can be proved using the Riemann–Roch theorem for surfaces.

EXERCISE 2.17. Let $S \subset \mathbb{P}^3$ be a smooth cubic surface over an algebraically closed field and let $\{L_i : i = 1, \dots, 27\}$ the set of twenty-seven lines on it. Let $D \subset S$ be any effective divisor. Then

1. D is linearly equivalent to $\sum a_i L_i$ for some $a_i \in \mathbb{Z}$.
2. Two divisors on a cubic surface are linearly equivalent if and only if they have the same intersection number with every line.
3. D is linearly equivalent to $\sum a_i L_i$ for some $a_i \in \mathbb{N}$.

Specific nonrational cubic surfaces over $\mathbb{Q}$ are constructed in the next exercise.

EXERCISE 2.18. 1. Find all lines on a smooth cubic surface given by an equation of the form $u^3 = f_3(x, y)$ in affine coordinates. In particular, find the lines on the Fermat hypersurface $a_1x_1^3 + a_2x_2^3 + a_3x_3^3 = a_0$. Do the same for $u^2 = f_3(x, y)$.

2. Show that if a is a rational number that is not a perfect cube, then the cubic surface defined by $x_1^3 + x_2^3 + x_3^3 = a$ has Picard number one over $\mathbb{Q}$. Conclude that such surfaces are not rational.

In fact, Segre showed that a surface over $\mathbb{Q}$ defined by the equation $a_0x_0^3 + a_1x_1^3 + a_2x_2^3 + a_3x_3^3 = 0$ has Picard number one if and only if, for all permutations σ of four letters, the rational number

$$\frac{a_{\sigma(0)}a_{\sigma(1)}}{a_{\sigma(2)}a_{\sigma(3)}}$$

is not a cube (Segre, 1951). The proof of Exercise 2.18 easily generalizes to yield this stronger result.

EXERCISE 2.19. Reduce the case of Theorem 2.16 when k is not perfect to the perfect case.

2.5 Birational self-maps of the plane

In this section, we investigate more thoroughly the structure of birational self-maps of the projective plane over an algebraically closed field. In particular, we prove the following theorem of Noether and Castelnuovo.

THEOREM 2.20. *Every birational self-map of the projective plane over an algebraically closed field is a composition of projective linear transformations and standard quadratic transformations*

$$(x_0 : x_1 : x_2) \dashrightarrow \left(\frac{1}{x_0} : \frac{1}{x_1} : \frac{1}{x_2}\right).$$

HISTORICAL REMARK 2.21. The statement is originally due to Max Noether (1870), but it has been reworked several times since then, most notably by Castelnuovo (1901), Nagata (1960), Iskovskikh (1979), and Corti (1995).

Let $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a birational map. It is given by a linear subsystem $\Gamma \subset |nH|$ of degree n curves in $\mathbb{P}^2$. Let $P_1, \ldots, P_r$ be the (infinitely near) base points of Γ and $m_1, \ldots, m_r$ the corresponding multiplicities. We may assume that $m_1 \geq m_2 \geq \cdots$. The starting point is Noether's inequality

$$m_1 + m_2 + m_3 > n. \tag{2.21.1}$$

This turns out to be a formal consequence of the inequalities in Exercise 2.8.

If the corresponding points P_1, P_2, P_3 are actual points of $\mathbb{P}^2$, then by a linear change of coordinates we can assume that these are $(1 : 0 : 0)$, $(0 : 1 : 0)$ and $(0 : 0 : 1)$. Composing ϕ with the standard quadratic transformation gives a new map $\phi' : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ which is given by a linear subsystem of $|(2n - m_1 - m_2 - m_3)H|$. We are done by induction since $2n - m_1 - m_2 - m_3 < n$.

It may happen, however, that some of the P_1, P_2, P_3 are infinitely near. In this case the choice of the quadratic transformation is not clear and it is not always possible to lower the degree. These cases require a very careful study. Instead of doing this directly, we follow a more roundabout way.

Our proof of Theorem 2.20 follows the general outline of Castelnuovo's proof, with hindsight coming from the Sarkisov program. This results in a leisurely meandering argument whose advantages are truly apparent only in the higher dimensional versions.

The argument naturally divides into two parts. The first part is conceptual: we prove a factorization of any self-map of $\mathbb{P}^2$ in terms of "elementary" birational maps between rational ruled surfaces. This point of view is generalized to dimension three by the Sarkisov program (Corti, 1995). The second part of the proof is a case by case analysis, in which the elementary maps between these rational surfaces are first assembled into de Jonquières maps (Definition 2.28), and subsequently broken down into standard quadratic transformations.

2.22. Rational ruled surfaces We begin by recalling the definition and basic properties of rational ruled surfaces, also known as rational surface scrolls. For more details, the reader is referred to Reid (1997, §2).

Definition 2.23. Fix a natural number q. The *rational ruled surface* $\mathbf{F}_q$ is the $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ given as the projectivized bundle of the rank two vector bundle $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(q)$. That is,

$$\mathbf{F}_q = \mathbb{P}(\mathcal{E}) = \mathrm{Proj}_{\mathbb{P}^1}(\mathrm{Sym}(\mathcal{E}))$$

where $\mathrm{Sym}(\mathcal{E})$ denotes the symmetric algebra of $\mathcal{E}$ over $\mathcal{O}_{\mathbb{P}^1}$.

In the special case where $q = 0$, the ruled surface $\mathbf{F}_0$ is simply the trivial fiber bundle $\mathbb{P}^1 \times \mathbb{P}^1$ (although technically speaking, the notation $\mathbf{F}_0$ indicates that we have chosen one of the projections to $\mathbb{P}^1$). In the case $q = 0$, let $B \subset \mathbf{F}_0$ denote any section with self intersection 0.

In the special case where $q = 1$, the rational ruled surface $\mathbf{F}_1$ is isomorphic to $\mathbb{P}^2$ blown up at a point. Let B denote the exceptional curve, a -1-curve. More generally, $\mathbf{F}_q$ is isomorphic to the blowup of the vertex of the cone over the rational normal curve of degree q in $\mathbb{P}^q$. Let B be the exceptional curve of this blowup. If $q > 0$ then B is the unique negative section of the projection $\mathbf{F}_q \to \mathbb{P}^1$.

The Picard group of the rational ruled surface $\mathbf{F}_q$ is the free Abelian group generated by the classes

$$A = \text{the class of a fiber}$$
$$B = \text{the class of the negative section.}$$

The intersection pairing is given by the rules

$$A^2 = 0 \qquad AB = 1 \qquad B^2 = -q.$$

Note also that the canonical class of $\mathbf{F}_q$ is given by

$$K_{\mathbf{F}_q} = -(2+q)A - 2B.$$

We prove Theorem 2.20 in steps. The first step, Theorem 2.24 below, shows that a birational self-map of the projective plane can be broken down into a composition of elementary "links." Later, in Theorems 2.30 and 2.32, we show that each link factors as needed into standard quadratic transformations.

THEOREM 2.24. *Every birational self-map of the projective plane is a composition of the following elementary maps or "links":*

1. *the involution* $\tau : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ *which exchanges the two factors;*
2. *the blowup* $\varepsilon : \mathbf{F}_1 \to \mathbb{P}^2$ *of a point* $P \in \mathbb{P}^2$*;*
3. *the inverse* $\varepsilon^{-1} : \mathbb{P}^2 \dashrightarrow \mathbf{F}_1$ *of a point blowup;*
4. *an elementary transformation* $\alpha_P : \mathbf{F}_q \dashrightarrow \mathbf{F}_{q\pm1}$*, by which we mean the blowing up a point* $P \in \mathbf{F}_q$*, followed by the contraction of the birational transform of the fiber through* P.

Note that the map described in (4) above is meaningful: blowing up the point P on $\mathbf{F}_q$, the birational transform A' of the fiber A through P is the -1-curve $\pi^*A - E$. Thus its blow-down is a smooth surface, which is $\mathbf{F}_{q+1}$ if P lies on the negative section B (or $q = 0$) and $\mathbf{F}_{q-1}$ otherwise.

In proving Theorem 2.24, we actually describe an algorithm for constructing such a factorization. In fact, it is convenient to consider more generally a birational map

$$\phi : \mathbf{F} \dashrightarrow \mathbb{P}^2$$

where F is either $\mathbb{P}^2$ or some rational ruled surface $\mathbf{F}_q$, for $q \geq 0$. We show that such a rational map factors into such links by induction on the *Sarkisov degree.*

To define the Sarkisov degree, first note that a rational map $\mathbf{F} \dashrightarrow \mathbb{P}^2$ is defined by a mobile linear system Γ such that either

(a) in case $\mathbf{F} = \mathbb{P}^2$, $\Gamma \subset |nH|$ for some $n \geq 1$, where H is a hyperplane section; or
(b) in case $\mathbf{F} = \mathbf{F}_q$, $\Gamma \subset |aA + bB|$ for some $b \geq 1$ and $a \geq bq$.

Indeed, in case (b) above, note that $B \cdot (aA + bB) = a - bq$. So if $a < bq$, B would be a base curve of Γ. With this notation, we can make the following definition.

DEFINITION 2.25. The *Sarkisov degree* of a rational map $\mathbf{F} \dashrightarrow \mathbb{P}^2$ given by the mobile linear system Γ is defined as

(a) $n/3$ in case $\mathbf{F} = \mathbb{P}^2$ and $\Gamma \subset |nH|$; or
(b) $b/2$ in case $\mathbf{F} = \mathbf{F}_q$ and $\Gamma \subset |aA + bB|$. (Note that if $\mathbf{F} = \mathbb{P}^1 \times \mathbb{P}^1$, the Sarkisov degree is only defined in terms of a choice of one of the two projections $\mathbf{F} \to \mathbb{P}^1$.)

We denote the Sarkisov degree of a rational map ϕ by s-deg(ϕ).

The aim of the Sarkisov degree is to compare a linear system to the canonical class. If $\mathbf{F} = \mathbb{P}^2$ then $-K_{\mathbf{F}} = 3H$ and so Γ is linearly equivalent to the Sarkisov degree times $-K_{\mathbf{F}}$.

In the second case $-K_{\mathbf{F}_q} = 2B + (2+q)A$. Then Γ is no longer linearly equivalent to a multiple of $-K_{\mathbf{F}_q}$, but

$$\Gamma \sim \text{s-deg}(\phi) \cdot (-K_{\mathbf{F}_q}) + (\text{a multiple of } A).$$

Any multiple of $|A|$ pulls back from $\mathbb{P}^1$, so we consider these "negligible." Admittedly, this is not very convincing for $q = 0$, but this is still the right thing to do.

The notion of Sarkisov degree is a subtle measure of the complexity of a birational map. We use it to prove Theorem 2.24 roughly as follows. Given a birational map $\phi\colon \mathbf{F} \dashrightarrow \mathbb{P}^2$, we find a procedure for constructing a link (of the type described in Theorem 2.24(1)–(4)) such that the composition $\phi \circ \alpha^{-1}$ is simpler than ϕ. We can then factor ϕ as

$$\mathbf{F} \overset{\alpha}{\dashrightarrow} \mathbf{F}' \overset{\phi\circ\alpha^{-1}}{\dashrightarrow} \mathbb{P}^2,$$

where the map $\phi \circ \alpha^{-1}$ is simpler than ϕ. Ideally, what is meant by "simpler" here is that the Sarkisov degree has dropped, in which case we would be done by induction. The proof proceeds by finding α such that the Sarkisov degree does not increase. In the case where it remains constant, we need to find secondary invariants that decrease. First, however, we need to know that there are base points of relatively high multiplicity.

Lemma 2.26. *Let $\phi : \mathbf{F} \dashrightarrow \mathbb{P}^2$ be a birational map given by some mobile linear system Γ, and assume that ϕ is not an isomorphism. Then Γ has a base point of multiplicity strictly greater than the Sarkisov degree of ϕ except in the following two cases:*

1. $\mathbf{F} = \mathbf{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$, *and* $\Gamma \subset |aA + bB|$ *for* $a < b$*; or*
2. $\mathbf{F} = \mathbf{F}_1$ *and* $\Gamma \subset |aA + bB|$ *where* $a/3 < b/2$.

In case (1), Γ *has a base point of multiplicity greater than* $a/2$.

Proof. We consider first the case where $\mathbf{F} = \mathbb{P}^2$. In this case, $\Gamma \subset |nH|$ for some $n \geq 1$, and we must show that Γ has a base point of multiplicity greater than $n/3$. Let $P_1, \dots, P_r$ be the base points of Γ, including the infinitely near ones, and let $m_1, \dots, m_r$ be their multiplicities. According to Exercise 2.8, we have

$$\sum m_i^2 = \Gamma^2 - 1 = n^2 - 1$$

$$\sum m_i = -K_{\mathbb{P}^2} \cdot \Gamma - 3 = 3n - 3.$$

If all $m_i \leq n/3$, then

$$n^2 - 1 = \sum m_i^2 \leq \frac{n}{3} \sum m_i = n^2 - n,$$

a contradiction because n is greater than one (as ϕ is not an isomorphism). This contradiction ensures that at least one $m_i > n/3$.

The case of a map $\phi \colon \mathbf{F}_q \dashrightarrow \mathbb{P}^2$ is only slightly more involved. Again using Exercise 2.8, we compute that

$$\sum m_i^2 = \Gamma^2 - 1 = b(2a - qb) - 1,$$

$$\sum m_i = -K_F \cdot \Gamma - 3 = 2(a + b) - qb - 3.$$

If all $m_i \leq b/2$, then

$$b(2a - qb) - 1 = \sum m_i^2 \leq \frac{b}{2} \sum m_i = b(a + b) - \frac{qb^2}{2} - \frac{3b}{2}.$$

In any case, $b \geq 1$, so $1 - (3/2)b$ is negative. Hence, $2a - qb < a + b - (qb/2)$ which, after rearranging, becomes

$$a < \left(\frac{q}{2} + 1\right) b.$$

Given that $qb \leq a$, it follows that $q < (q/2) + 1$. This is possible only when $q = 0$ or $q = 1$. In the former case, it follows that $a < b$, and in the latter case that $a/3 < b/2$.

Assume that we are in case (1). How does one find a base point of multiplicity greater than $a/2$? One way is to work through the above estimates. The slick way is to note that we can interchange the two copies of $\mathbb{P}^1$ and we can view the situation as $\Gamma \subset |bA + aB|$. By (1), if this does not have a base point of multiplicity greater than $a/2$ (which is now the Sarkisov degree), then $b < a$. However we already know that $a < b$, a contradiction. $\square$

The next lemma is the heart of the proof of Theorem 2.24.

LEMMA 2.27. *Let $\phi\colon \mathbf{F} \dashrightarrow \mathbb{P}^2$ be a rational map given by a mobile linear system Γ, where $\mathbf{F}$ is either $\mathbb{P}^2$ or a rational ruled surface $\mathbf{F}_q$. Suppose that Γ has a base point P of multiplicity greater than the Sarkisov degree of ϕ.*

1. *If $\mathbf{F} = \mathbb{P}^2$, then the Sarkisov degree of $\phi \circ \varepsilon$ is strictly less than the Sarkisov degree of ϕ, where $\varepsilon : \mathbf{F}_1 \to \mathbb{P}^2$ is the blowup of $\mathbb{P}^2$ at P.*
2. *If $\mathbf{F} = \mathbf{F}_q$, then the Sarkisov degree of ϕ is equal to the Sarkisov degree of $\phi \circ \alpha_P^{-1}$, where $\alpha_P : \mathbf{F}_q \dashrightarrow \mathbf{F}_{q\pm1}$ is the elementary transformation described in Theorem 2.24(4).*

PROOF. This is a consequence of Lemma 2.26. For instance, consider first the case when $\mathbf{F} = \mathbb{P}^2$ and the map ϕ is given by a linear system $\Gamma \subset |nH|$ having a basepoint P of multiplicity $m > \frac{n}{3}$. Letting $\varepsilon : \mathbf{F}_1 \to \mathbb{P}^2$ be the blowup of P, note that the composition

$$\mathbf{F}_1 \xrightarrow{\varepsilon} \mathbb{P}^2 \overset{\phi}{\dashrightarrow} \mathbb{P}^2$$

is defined by a linear system Γ' contained in $|n(A + B) - mB| = |nA + (n - m)B|$. Indeed, since ε is given by $|A + B|$, the hyperplane section of $\mathbb{P}^2$ pulls back to $|A + B|$ under ε. Furthermore, B is the exceptional fiber over the base point P of multiplicity n, so the fixed component of the pullback of Γ is mB. Because we are assuming that $m > n/3$,

$$\Gamma' \subset |nA + (n - m)B|$$

implies that the Sarkisov degree of Γ' is $\frac{n-m}{2}$, which is strictly less than $\frac{n}{3}$.

Similarly, consider the case where $\mathbf{F} = \mathbf{F}_q$ and the map ϕ is given by a linear system Γ contained in $|aA + bB|$ having a base point of multiplicity $m > \frac{b}{2}$. In this case the composition

$$\mathbf{F}_{q\pm1} \xrightarrow{\alpha_P^{-1}} \mathbf{F}_q \overset{\phi}{\dashrightarrow} \mathbb{P}^2$$

is defined by a linear system Γ' contained in $|(a + b - m)A + bB|$ if $P \in B$, or in $|(a - m)A + bB|$ otherwise. In either case, the Sarkisov degree is unchanged. $\square$

Proof of Theorem 2.24. We describe an algorithm for factoring any birational map $\phi : \mathbf{F} \dashrightarrow \mathbb{P}^2$ as a composition of the desired links. Suppose that ϕ is given by the mobile linear system Γ.

Begin by checking whether Γ has base points of multiplicity greater than the Sarkisov degree of ϕ. If not, then by Lemma 2.26, $\mathbf{F}$ is either $\mathbf{F}_0$ or $\mathbf{F}_1$. If $\mathbf{F} = \mathbf{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$, then $a < b$ by Lemma 2.26(1). Thus, if we compose ϕ with the link τ interchanging the factors, the Sarkisov degree drops from $b/2$ to $a/2$. If $\mathbf{F} = \mathbf{F}_1$, then consider $\phi \circ \varepsilon^{-1}$, where ε is the link $\mathbf{F}_1 \to \mathbb{P}^2$ contracting the negative section. Its Sarkisov degree is $a/3$. By Lemma 2.26(2), this is strictly smaller than $b/2$, the Sarkisov degree of ϕ.

If ϕ has a base point of multiplicity m, higher than the Sarkisov degree, there are two cases to consider.

If $\mathbf{F} = \mathbb{P}^2$, then we blow up a base point of maximum multiplicity. $\phi \circ \varepsilon$ is given by a linear subsystem of $|nA + (n-m)B|$, which has Sarkisov degree $(n-m)/2 < n/3$ since $m > n/3$.

If $\mathbf{F} = \mathbf{F}_q$, then blow up a point P of maximum multiplicity $m > b/2$. By Lemma 2.27, we can factor ϕ as

$$\mathbf{F}_q \overset{\alpha_P}{\dashrightarrow} \mathbf{F}_{q\pm 1} \overset{\phi\alpha_P^{-1}}{\dashrightarrow} \mathbb{P}^2$$

where α_P is the elementary transform as described in Theorem 2.24(4) and the Sarkisov degree of $\phi \circ \alpha_P{}^{-1}$ is equal to the Sarkisov degree of ϕ. In this case, we can not induce on the Sarkisov degree; instead we induce on the sum of the multiplicities of the base points of Γ, including the infinitely near ones. For this, it suffices to show that the sum of the multiplicities of the base points of the mobile linear system Γ' describing the map $\phi \circ \alpha_P{}^{-1}$ is strictly less than the sum of the multiplicities of the base points of Γ.

When we blow up the point P, we remove a base point of multiplicity m. After this, we contract the birational transform A' of the fiber A through P. The intersection number of A with Γ is b, so the intersection number of A with Γ' is $b - m$. When we contract A', we introduce a new base point with multiplicity $b - m$. Thus α_P exchanges the multiplicity m point P for a multiplicity $b - m$ point, and leaves the other base points unchanged. Since $m > b/2$, this lowers the sum of the multiplicities. This concludes the proof of Theorem 2.24. □

In order to prove the theorem of Castelnuovo and Noether, we must therefore understand how to relate the links of Theorem 2.24 to quadratic maps of $\mathbb{P}^2$. We do this by first decomposing into de Jonquières maps.

Definition 2.28. A birational map $\psi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is de Jonquières if one of the following equivalent conditions holds.

1. After composing with a suitable linear transformation $\ell\colon \mathbb{P}^2 \to \mathbb{P}^2$, the map $\psi' = \ell\psi$ factors as
$$\mathbb{P}^2 \overset{\varepsilon^{-1}}{\dashrightarrow} \mathbf{F}_1 \overset{\alpha}{\dashrightarrow} \mathbf{F}_1 \overset{\varepsilon}{\dashrightarrow} \mathbb{P}^2,$$
where α is a *square* birational map, that is, α commutes with the structure morphism $\mathbf{F}_1 \to \mathbb{P}^1$.
2. There is a pencil of lines P whose birational transform by ψ is also a pencil of lines.
3. The map ψ is defined by a linear system Γ of plane curves of degree n, with "characteristic" $(n-1, 1, \dots, 1)$. This means that Γ has a unique base point Q with multiplicity $n-1$ and the remaining base points (necessarily $2n-2$ of them, possibly infinitely near) all have multiplicity one.

To see that the conditions of Definition 2.28 are equivalent, note that a map ψ satisfying the second condition restricts to a linear map on each line through Q. Thus ψ can be viewed as a birational map of the scroll $\mathbf{F}_1$ obtained by blowing up $Q \in \mathbb{P}^2$. □

EXERCISE 2.29. Show that, up to linear changes of coordinates, the following are the only quadratic transformations of $\mathbb{P}^2$.

1. The map $(x_0 : x_1 : x_2) \dashrightarrow (x_1x_2 : x_0x_2 : x_0x_1)$, called the standard quadratic map;
2. The map $(x_0 : x_1 : x_2) \dashrightarrow (x_0x_2 : x_1x_2 : x_0^2)$;
3. The map $(x_0 : x_1 : x_2) \dashrightarrow (x_0^2 : x_0x_1 : x_1^2 + x_0x_2)$.

Note that any quadratic transformation $\mathbb{P}^2$ is de Jonquières, because any base point of a system of quadrics has multiplicity one.

THEOREM 2.30. *Every birational self-map of the projective plane is a composition of de Jonquières transformations.*

This is an immediate consequence of the following more precise statement, due to Castelnuovo (1901):

PROPOSITION 2.31. *Given any birational self-map ϕ of the projective plane, there is a de Jonquières map ψ such that the Sarkisov degree of $\phi \circ \psi^{-1}$ is strictly less than the Sarkisov degree of ϕ.*

PROOF. Recall the factorization process of the proof of Theorem 2.24. We begin with a blowup $\varepsilon^{-1} : \mathbb{P}^2 \dashrightarrow \mathbf{F}_1$ and proceed with a chain of elementary transformations $\mathbf{F}_q \dashrightarrow \mathbf{F}_{q\pm 1}$ of rational ruled surfaces. Eventually, we either reach a blow-down $\varepsilon\colon \mathbf{F}_1 \to \mathbb{P}^2$, in which case we have just completed a de Jonquières map with the required property, or we reach $\mathbf{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$, and there is a point $P \in \mathbf{F}_0$ with multiplicity m satisfying $a/2 < m \leq b/2$. In the latter case, the algorithm would proceed by "switching factors" of $\mathbb{P}^1 \times \mathbb{P}^1$ and then continue with the elementary transformation centered at P; we don't want to do this, we want to create a de Jonquières map. The map $\mathbf{F}_0 \dashrightarrow \mathbb{P}^2$ induced by ϕ is defined by a linear system $\subset |aA + bB|$. Let us not switch the factors, and go on instead with an elementary transformation $\mathbf{F}_0 \dashrightarrow \mathbf{F}_1$ centered at P, and follow that with the blow-down $\mathbf{F}_1 \to \mathbb{P}^2$. We have created a de Jonquières map $\psi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ which is the composition of all these maps:

$$\mathbb{P}^2 \dashrightarrow \mathbf{F}_0 \dashrightarrow \mathbf{F}_1 \to \mathbb{P}^2.$$

It is easy to see that the map $\mathbf{F}_1 \dashrightarrow \mathbb{P}^2$ induced by ϕ is defined by a linear system contained in $|(a+b-m)A + bB|$, hence

$$\text{s-deg}(\phi \circ \psi^{-1}) = \frac{a+b-m}{3} < b/2.$$

$\square$

THEOREM 2.32. *Every de Jonquières transformation is the composite of linear maps and standard quadratic transformations.*

PROOF. We follow Nagata (1960). First we prove a result on the factorization of a square birational map

$$\alpha\colon \mathbf{F}_1 \dashrightarrow \mathbf{F}_1.$$

Let $P \in \mathbf{F}_1$ be a point not on the -1-curve B, and let Q be another point, possibly infinitely near to P (i.e. lying on the blowup of $P \in \mathbf{F}_1$). The composition

$$\alpha_{QP} = \alpha_Q \alpha_P : \mathbf{F}_1 \dashrightarrow \mathbf{F}_1$$

is a birational transformation of $\mathbf{F}_1$. We now claim that:

CLAIM 2.33. α is a composition of transformations of the type α_{QP}.

The claim is proved by decreasing induction on a discrete invariant which we now define. Let P be a point of $\mathbf{F}_1$, possibly infinitely near. This means that $P \in S$ is an honest point of a surface $S \to \mathbf{F}_1$ obtained from $\mathbf{F}_1$ by a sequence of blowups of smooth points; we define the *level* $h(P)$ of P to be the minimum number of blowups needed. In particular, $h(P) = 0$ if and only if $P \in \mathbf{F}_1$. We say that an infinitely near point $P \in S$ does not lie on a curve B if P is not contained in the birational transform of B on S.

Suppose that $\alpha : \mathbf{F}_1 \dashrightarrow \mathbf{F}_1$ is square, that is, that it maps a general fiber to a general fiber. Then since ε maps a general fiber to a line, the composition

$$\mathbf{F}_1 \overset{\alpha}{\dashrightarrow} \mathbf{F}_1 \overset{\varepsilon}{\dashrightarrow} \mathbb{P}^2$$

is given by a linear system Γ contained in $|aA + B|$ for some positive a.

Such linear systems have many pleasant properties. Let P be a base point of multiplicity m_P and A a fiber through P. Then $m_P \le (A \cdot \Gamma) = 1$, so all base points have multiplicity one. The total number of base points is $2a - 2$. If we blow up P, the birational transform A' of A has 0 intersection number with Γ', so A' is contracted by Γ'. Similarly, the number of base points (including infinitely near ones) on B is at most $(B \cdot \Gamma) = a - 1$. If $a > 1$ then we see that not all base points are on B.

If $a = 1$ then α is an isomorphism. If $a > 1$, we induce on lexicographically ordered pairs (r, h), where r is the number of base points of the linear system Γ, including the infinitely near ones and h is the minimum level of a base point not lying on B.

If $h = 0$, there is a base point $P \in \mathbf{F}_1$ not on B. Let Q be another base point, possibly infinitely near to P. Now one readily checks that that $\alpha \circ \alpha_{QP}^{-1}$ has fewer base points than α. Thus the invariant (r, h) has dropped (although it is possible that h has increased).

If $h > 0$, take a sufficiently general point $P \in \mathbf{F}_1$ (not a base point), and let $Q \in B \subset \mathbf{F}_1$ be the center on $\mathbf{F}_1$ of a base point with minimal level. Then one easily checks that the birational map $\alpha \circ \alpha_{QP}^{-1}$ has invariant $(r, h - 1)$. Thus the invariant has again dropped.

Because we have given a recipe for factoring α as a composition of a map of the form α_{QP} followed by a map with strictly smaller invariant (r, h), the proof of the claim is complete by induction.

Finally, we conclude by showing that the claim implies the theorem. It suffices to show that

$$\varepsilon \circ \alpha_{QP} \circ \varepsilon^{-1} \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$$

is the composition of standard quadratic maps, where as usual $\varepsilon \colon \mathbf{F}_1 \dashrightarrow \mathbb{P}^2$ is the blow-down of the negative section. There are three cases to consider.

Case 1: Q is not infinitely near to P and $Q \notin B$. In this case, we readily verify that $\varepsilon \circ \alpha_{QP} \circ \varepsilon^{-1}$ is a standard quadratic map.

Case 2: Q is not infinitely near to P and $Q \in B$. In this case, $\varepsilon \circ \alpha_{QP} \circ \varepsilon^{-1}$ is a quadratic map as in Exercise 2.29(2). Choose a general point R and write

$$\varepsilon \circ \alpha_{QP} \circ \varepsilon^{-1} = (\varepsilon \circ \alpha_Q \circ \alpha_R \circ \varepsilon^{-1}) \circ \left(\varepsilon \circ \alpha_R^{-1} \circ \alpha_P \circ \varepsilon^{-1}\right)$$

the composition of two standard quadratic maps.

Case 3: Q is infinitely near to P. As in case 2, $\varepsilon \circ \alpha_{QP} \circ \varepsilon^{-1}$ is a quadratic map as in Exercise 2.29(2). □

For the final exercise of this chapter, we outline a different proof that every de Jonquières map is a composition of quadratic transformations.

EXERCISE 2.34. 1. Show that, after choosing suitable affine coordinates on the source and target, every de Jonquières map can be written as

$$\psi : (x, y) \to \left(x, \frac{a(x)y + b(x)}{c(x)y + d(x)} \right)$$

where $a(x), b(x), c(x), d(x) \in k(x)$ are rational functions on $\mathbb{P}^2$ such that $a(x)d(x) - b(x)c(x) \neq 0$.

2. Using (1), show that every de Jonquières map is a composition of quadratic maps.
3. Show that every de Jonquières map is a composition of standard quadratic maps.

3
Rational surfaces

We have already seen that the projective line stands alone among all curves: any projective curve remotely behaving like $\mathbb{P}^1$ is in fact isomorphic to $\mathbb{P}^1$. By contrast, there are numerous surfaces that share various features of the projective plane. The oldest example is the smooth quadric surface, although cubic surfaces were also recognized classically as being similar to the projective plane. Over an algebraically closed field, quadrics and cubics are birationally equivalent to the plane, and this explains most of the similarities.

In this chapter, we develop these examples into a systematic theory of surfaces sharing numerical invariants with the projective plane. The first major result, the rationality criterion of Castelnuovo, completely classifies the geometrically rational surfaces as those for which certain simple numerical invariants are the same as the plane's.

Classically, the rationality criterion of Castelnuovo was viewed as one of the cornerstones of the theory of surfaces. More recently, algebraic geometers have come to view the theory of minimal models and the study of Del Pezzo surfaces, which are a nice class of geometrically rational varieties, as the two pillars. This point of view has lessened the importance of the Castelnuovo criterion, which instead appears as a nice consequence of minimal model theory. Indeed, whereas Castelnuovo's rationality criterion seems to have no good analog in higher dimensions, the minimal model program and the study of Fano varieties (higher dimensional analogs of Del Pezzo surfaces), are expected to hold the key to classifying birational equivalence classes of varieties of any dimension. In retrospect, the theory of minimal models of higher dimensional varieties is a clear generalization of the surface theory, but it took almost a hundred years to understand this. Here we treat only the surface case, referring the interested reader to Kollár and Mori (1998) for the higher dimensional case.

In Section 1, we discuss Castelnuovo's rationality criterion for surfaces over an algebraically closed field. In Section 2, we place it in the context of the minimal model program, which also provides a way to treat varieties over algebraically non-closed fields. The proof of Castelnuovo's criterion and these related results over arbitrary fields is carried out in the third section. Here, we give a direct geometric proof using classical methods. However, the formulation of the key result, Theorem 3.5, is in the spirit of the minimal model program.

Section 4 contains a long overdue discussion of the question "When can we define a variety with equations whose coefficients are in a given field?" Nowadays this is usually arrived at as a minor consequence of the theory of Hilbert schemes, but the original methods of Weil give better results quickly and concretely.

The theory of Del Pezzo surfaces is taken up in Section 5. The main result is Theorem 3.36, giving a complete description of all low degree Del Pezzo surfaces.

3.1 Castelnuovo's rationality criterion

One of the first questions of classical surface theory is how to characterize the surfaces behaving like the projective plane: what features of the projective plane actually characterize its birational equivalence class? After a treatment of many examples by Clebsch, Noether and Bertini, a complete answer was found by Castelnuovo:

THEOREM 3.1. *A smooth projective surface over an algebraically closed field is rational if and only if it has no holomorphic one-forms and it admits no bicanonical curves.*

The bicanonical linear system on a surface S is the linear system $|2K_S|$, thus to admit no bicanonical curves means that this linear system is empty or, equivalently, that the second plurigenus is zero.

Theorem 3.1 is a strong converse of Theorem 1.52 stating that a smooth projective rational variety admits no global sections of any tensor power of its sheaf of regular differential one-forms. Indeed, the vanishing of the irregularity is the $m = 1$ case of the vanishing of $\Gamma(S, \Omega_S^{\otimes m})$ and the nonexistence of bicanonical curves is a consequence of the $m = 4$ case. To see this, note that there is an embedding $\mathcal{O}_S(2K_S) \hookrightarrow \Omega_S^{\otimes 4}$ obtained as the square of the natural injection

$$\mathcal{O}_S(K_S) \cong \wedge^2 \Omega_S \hookrightarrow \Omega_S^{\otimes 2}.$$

(This natural injection is the "determinant map," sending $w \wedge v$ to $w \otimes v - v \otimes w$.)

Over $\mathbb{C}$, from Hodge theory, the vanishing of $H^0(S, \Omega_S)$ is equivalent to the vanishing of $H^1(S, \mathcal{O}_S)$; see Griffiths and Harris (1978, 0.6–7). We restate Castelnuovo's rationality criterion in the following form:

THEOREM 3.2. *A smooth projective surface S over an algebraically closed field is rational if and only if*

$$H^1(S, \mathcal{O}_S) = 0 \quad \textit{and} \quad H^0(S, \mathcal{O}_S(2K_S)) = 0.$$

In proving Castelnuovo's rationality criterion, we are led to consider special curves of arithmetic genus zero on the surface S. Their basic properties are summarized here for future reference.

3.3. ARITHMETIC GENUS OF CURVES ON SURFACES. The arithmetic genus of a connected curve C is defined as

$$p_a(C) = \dim H^1(C, \mathcal{O}_C).$$

Of course, if the curve is smooth, then this agrees with the usual notion of (geometric) genus. On the other hand, it is not hard to see that every reduced and irreducible curve of arithmetic genus zero over an algebraically closed field is a smooth rational curve. Further properties of curves of arithmetic genus zero are treated in Exercise 3.4 below.

The classical adjunction formula for a smooth curve on a smooth surface can be generalized to singular curves with the arithmetic genus playing the role of the usual geometric genus. Indeed, if C is an arbitrary effective divisor on a smooth surface S, then the adjunction formula states that

$$2p_a(C) - 2 = C \cdot (C + K_S); \tag{3.3.1}$$

see Shafarevich (1994, VI.1.4) or Hartshorne (1977, V Exercise 1.3). We make much use of this adjunction formula in this chapter.

EXERCISE 3.4. (a) Prove that a reduced and connected curve C on a smooth surface over an algebraically closed field has arithmetic genus zero if and only if the following three conditions on C are satisfied.

1. Every irreducible component of C is a smooth rational curve.
2. The only singular points of C are transverse intersections of two components, and any two components intersect in at most one point.
3. The dual graph of C is a tree, where the dual graph is that whose vertices are indexed by the irreducible components of C, with two vertices connected by as many edges as the intersection number of the two components.

(b) Let L be a line bundle on a reduced curve C of arithmetic genus zero, and assume that L has non-negative degree on every irreducible component of

C. Show that L is generated by its global sections, that $H^1(C, L) = 0$, and that the dimension of $H^0(C, L)$ is equal to the degree of L plus the number of connected components of C.

(c) Let D be a reduced irreducible geometrically connected curve of arithmetic genus zero on a smooth surface defined over a perfect field k. Prove that $D_{\bar{k}}$ is either irreducible or it has two irreducible components which are then conjugates of each other. Every such D is isomorphic to a plane conic.

3.2 Minimal models of surfaces

We have seen that rationality is not an easy condition to check. Thus it is desirable to have a method which to some extent automates this task. The first part of such a program is given by the theory of minimal models of surfaces.

Consider a smooth projective surface S over an algebraically closed field. Let P be any point on S and let $S' \to S$ denote the blowup of P. The exceptional divisor of this blowup is a smooth projective rational curve E whose self-intersection number E^2 is -1.

The contractibility criterion of Castelnuovo asserts that, conversely the existence of such a curve characterizes blowups. More precisely, if a surface contains a curve isomorphic to a projective line and having self-intersection -1, then that curve can be blown down to a smooth point on another surface. Such a curve is called a -1-*curve*.

For the proof of this fact see Hartshorne (1977, V.5.7) or Barth *et al.* (1984, III.4).

This contractibility criterion provides a method for simplifying smooth projective surfaces as follows. Given a surface $T = T_0$, we first check whether it contains any -1-curves (in practice, this may not be easy). If so, pick one of them and blow it down. This gives us a birational morphism $T_0 \to T_1$. Now we look for -1-curves on T_1 and continue. In this way we obtain a sequence of smooth projective surfaces

$$T_0 \to T_1 \to T_2 \to \cdots \to T_m.$$

With each blow-down, the rank of the Picard group of T_i drops by one. Thus, because the Picard group has finite rank, the procedure must eventually stop. (Over $\mathbb{C}$, the finiteness of the Picard number follows easily from topological considerations, see Griffiths and Harris (1978, 1.1). In positive characteristic this is quite a bit harder, see Kleiman (1972).)

The final surface T_m has no -1-curves. A surface containing no -1-curves is said to be *minimal*; we also say that T_m is a *minimal model of* T. It turns out that in most cases, T_m is unique. The exception is when we are dealing with the birational equivalence class of rational or ruled surfaces, exactly the cases in which we are most interested. See for example Barth *et al.* (1984, III.4).

Using this "simplification" procedure, any birational question about smooth projective surfaces can be reduced, in principle, to a question about minimal surfaces. In particular, since both the irregularity and the plurigenera are birational invariants, Castelnuovo's rationality criterion, Theorem 3.2, is implied by the following stronger statement.

THEOREM 3.5. *A smooth minimal surface over an algebraically closed field which has irregularity zero and admits no bicanonical curves is isomorphic either to the projective plane or to a rational ruled surface.*

This theorem can fail over a non-algebraically closed field, as does Castelnuovo's criterion, as shown by a cubic surface of Picard number one (see Section 2.4). It turns out, however, that there are results analogous to Theorem 3.5 valid over arbitrary fields. To state them, we need to understand minimal models over algebraically non-closed fields. Because we prefer to give nice geometric arguments, we restrict attention to perfect fields, although the results we prove are valid over an arbitrary field. See Mori (1982) or Kollár (1996, Ch. 3).

3.6. DIGRESSION ON NONPERFECT FIELDS. Consider a variety X defined over a fixed ground field k. Let P be a closed (but not necessarily k-rational) point of X. Technically speaking, the point P corresponds to some maximal ideal in a local coordinate ring for X. Often, it is more convenient to think of P as a Galois orbit of points $P_1, \dots, P_d$ defined over $\bar{k}$. However, this way of thinking is valid only when the ground field k is perfect.

For example, consider the points of the affine line $\mathbb{A}^1 = \operatorname{Spec} k[x]$. If $a_1, \dots, a_n \in \bar{k}$ are a complete set of conjugates, then

$$\prod_i (x - a_i) = \sum (-1)^i \sigma_i x^i$$

is a polynomial whose zero set is precisely the set $\{a_1, \dots, a_n\}$. The coefficients σ_i are the elementary symmetric polynomials in the a_i, hence they are invariant under the Galois group $\operatorname{Gal}(\bar{k}/k)$. If k is perfect then this implies that $\sigma_i \in k$, so the set $\{a_1, \dots, a_n\}$ is the zero set of an irreducible polynomial in $k[x]$. The corresponding closed point of $\mathbb{A}^1_k = \operatorname{Spec} k[x]$ corresponds to the maximal ideal of $k[x]$ generated by this polynomial.

If, however, k is not perfect, then the σ_i may not be in k. In this case the best we can do is to claim that there is a polynomial in $k[x]$ whose zero set is $\{a_1, \dots, a_n\}$, each with multiplicity p^m for some $m \geq 0$ where p is the characteristic of k. For instance, let a be an element of $\bar{k}$ that is not in k but whose pth power b is in k.

Then $x^p - b$ is an irreducible polynomial whose zero set is $\{a\}$ with multiplicity p. There is no polynomial in $k[x]$ whose zero set contains a with multiplicity less than p. This phenomenon leads to technicalities over nonperfect fields that we prefer to avoid here.

3.7. Minimal models of surfaces over perfect fields. Fix a perfect ground field k. Let S be a smooth projective surface over k, and let P be a closed (but not necessarily rational) point of S. As noted above, we can think of P as a Galois orbit of points $P_1, \dots, P_d \in S(\bar{k})$.

Let $S' \to S$ denote the blowup of P, and let E denote the exceptional curve. Thus $S'_{\bar{k}}$ is obtained from $S_{\bar{k}}$ by blowing up all the points $P_1, \dots, P_d$. So E is a reduced and irreducible curve over k, but $E_{\bar{k}}$ is a disjoint union of d copies of $\mathbb{P}^1$. In particular $E^2 = -d$ and $E \cdot K_{S'} = -d$. We call any such curve a -1-*curve*. We can also think of a -1-curve as a Galois orbit of disjoint -1-curves $E_1, \dots, E_d \subset S_{\bar{k}}$; see Corollary 3.30. Over an algebraically closed field k, this definition of -1-curve agrees with the classical one because in this case every Galois orbit has cardinality $d = 1$.

A surface is called *minimal over k* if it does not contain any -1-curves in the sense defined above. Equivalently, the surface S_k is minimal if $S_{\bar{k}}$ does not contain a Galois orbit of disjoint -1-curves. The reader is cautioned that $S_{\bar{k}}$ may well contain many -1-curves even if S does not contain any. Indeed, according to Theorem 2.16 any cubic surface of Picard number one has this property.

As before, the contractibility criterion of Castelnuovo holds. Thus, given a smooth projective surface T over k we obtain a sequence of smooth projective surfaces

$$T_0 \to T_1 \to T_2 \cdots \to T_m,$$

all defined over k, where the final surface T_m has no -1-curves.

The situation is similar over nonperfect fields; see Kollár and Mori (1998, 1.4). One must take into account that blowing up closed nonrational points may give unexpected results; we hint at some of these difficulties in the next exercise.

Exercise 3.8. Let u and v be indeterminates over a field k of prime characteristic p. Consider the affine plane $\mathbb{A}^2$ over the field $k(u, v)$.

(1) Let $P \in \mathbb{A}^2$ denote the closed point corresponding to $(0, \sqrt[p]{v})$. Prove that the blowup of the plane at this point is regular (that is, every local ring is regular), but that it has one nonsmooth point.
(2) Let $Q \in \mathbb{A}^2$ denote the closed point corresponding to $(\sqrt[p]{u}, \sqrt[p]{v})$. Prove that the blowup along Q is is regular but it is not geometrically normal along its exceptional curve.

EXERCISE 3.9. Let S be a smooth projective surface over a perfect field, and let C be an irreducible curve on S. Show that C is a -1-curve if and only if $C \cdot K_S < 0$ and $C^2 < 0$.

3.10. FACTORING BIRATIONAL MORPHISMS. A classical result of surface theory asserts that, over an algebraically closed field, every birational morphism between smooth surfaces is a composition of blowups of points; see, for example, Shafarevich (1994, IV.3).

In fact, this assertion remains true over any field. Over a perfect field, this is easy to see. Indeed, given a morphism $f : S \to T$ defined over k, the set Z of points of T where f^{-1} is not a morphism is defined over k, so it can be viewed as a union of Galois orbits of points on $T(\bar{k})$. Now, the standard proof of the factorization theorem over $\bar{k}$ requires that each of these points should be blown up in order to factor f over $\bar{k}$. But blowing up all these points is the same as blowing up Z. Thus f can be factored through the blowup of T along Z. The proof follows by induction as in the classical case.

3.11. CONIC BUNDLES. As we have seen, over a non-algebraically closed field, every geometrically rational curve is a plane conic. Thus it is reasonable to expect that similarly, the correct generalization of ruled surfaces should be *conic bundles*.

DEFINITION 3.12. A *conic bundle* is a smooth projective surface S together with a morphism to a smooth curve $S \to C$ such that every fiber is isomorphic to a (possibly singular) plane conic.

Basic properties of conic bundles are worked out in the next exercise.

EXERCISE 3.13. Let $f : S \to C$ be a conic bundle over an algebraically closed field.

1. Show that every fiber is either a smooth conic or a pair of intersecting lines (not a double line), and that the irreducible components of the singular fibers are -1-curves on S.
2. Show that S can be obtained from a $\mathbb{P}^1$-bundle $T \to C$ by blowing up one point in each of finitely many fibers. Conversely, any such surface is a conic bundle.
3. Show that $K_S^2 \leq 8(1 - g(C))$, where $g(C)$ denotes the genus of C, with equality if and only if every fiber is smooth.
4. Show that $f_*(\mathcal{O}(-K_S))$ is a rank 3 vector bundle on C and that there is a natural embedding

$$S \hookrightarrow \mathbb{P}(f_*(\mathcal{O}(-K_S))) = \mathrm{Proj}_C(\mathrm{Sym}(f_*(\mathcal{O}(-K_S))),$$

where S is realized as a family of conics in the projective plane fibers of $\mathbb{P}(f_*(\mathcal{O}(-K_S)))$.

5. Let T be a smooth projective surface and let $g : T \to C$ be a morphism with connected fibers. Show that T is a conic bundle over C if and only if $-K_T$ is g-ample.
6. Let T be a smooth projective surface over a field k and $g : T \to C$ a morphism whose generic fiber is a smooth rational curve. Assume that no -1-curve over k is contained in the union of finitely many fibers of g. Show that T is a conic bundle over C.

We are now ready to state the first structure theorem of rational–like surfaces over perfect fields; it is due to Iskovskih.

THEOREM 3.14 (Iskovskih, 1979). *Let S be a smooth projective geometrically irreducible minimal surface over a perfect field. Assume that the irregularity of S is zero and that S admits no bicanonical curves. Then S is isomorphic to one of the following:*

1. *the projective plane;*
2. *a quadric surface in $\mathbb{P}^3$;*
3. *a conic bundle over a geometrically rational curve;*
4. *a surface whose Picard group is generated by the canonical bundle.*

We have already seen that certain cubic surfaces satisfy (4) above. In Section 4, we systematically study surfaces whose Picard group is generated by the canonical class.

Theorem 3.14 will be proved in the next section. Here we show simply that Theorem 3.14 implies Castelnuovo's rationality criterion.

PROOF THAT THEOREM 3.14 IMPLIES 3.5. Suppose that S is a surface defined over an algebraically closed field satisfying the hypothesis of Theorem 3.5. By Theorem 3.14, then, S must be isomorphic to one of the four listed types of varieties. Since quadric surfaces are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and a minimal conic bundle is a $\mathbb{P}^1$ bundle over $\mathbb{P}^1$ by Exercise 3.13.2, we only have to consider the case where the surface S has Picard group generated by K_S.

In fact, we claim that such *minimal* surfaces do not exist over algebraically closed fields. This is actually a quite thorny problem of surface theory; no elementary geometric proof for this fact is known. Instead, we summarize two approaches, and the interested reader can refer to the literature for the details.

The first approach is to classify all surfaces whose Picard group is generated by K_S, and thereby observe that they all admit -1-curves. This is relatively easy to do using the results of Section 3 and some deformation theory. See, for example, Kollár (1996, III.2–3).

Over $\mathbb{C}$, there is an alternate argument, relying on topology and Hodge theory (see, for instance Griffiths and Harris (1978, 0.6–7) for the relevant facts). By assumption $H^{1,0}(S) = H^{0,1}(S) = 0$ and $H^{2,0}(S) = H^{0,2}(S) = 0$. Thus $\mathrm{Pic}(S) = H^2(S, \mathbb{Z})$. Since $\mathrm{Pic}(S) \cong \mathbb{Z}$, the topological Euler characteristic $e(S)$ must be three. From Noether's formula $K_S^2 + e(S) = 12\chi(\mathcal{O}_S)$ we conclude that $K_S^2 = 9$. (A relatively elementary proof of Noether's formula is given in Griffiths and Harris (1978, 4.6). Nowadays it is viewed as the surface case of a general formula of Hirzebruch expressing the holomorphic Euler characteristic in terms of Chern classes. See Hartshorne (1977, App. A).) On the other hand, Poincaré duality says that the intersection form on $H^2(S, \mathbb{Z})$ of a compact orientable 4-manifold is unimodular. Thus the generator K_S of $H^2(S, \mathbb{Z})$ should satisfy $K_S^2 = 1$, a contradiction. □

3.3 Rational surfaces over perfect fields

In this section, we prove Theorem 3.14 on the classification of smooth minimal surfaces over a perfect field whose irregularity and second plurigenus both vanish.

Let S be such a surface, and assume that its canonical class does not generate the Picard group. In order to show that S is isomorphic to one of the three types listed in Theorem 3.14(1–3), we need to find a suitable linear system of curves on S. Our strategy is to look for such curves among the irreducible components of the members of a linear system $|D + mK_S|$, where D is any curve not linearly equivalent to a multiple of K_S and m is a carefully chosen integer. We outline the proof of Theorem 3.14 in three steps.

First, Lemma 3.15 ensures that $|D + nK_S|$ is empty for large n, a fact that has classically been called *termination of adjunction*. This allows us to choose m largest possible such that $|D + mK_S|$ is not empty. Because of our choice of D, we know that $D + mK_S$ is not linearly trivial; thus the members of $|D + mK_S|$ are effective nontrivial divisors on S.

Next, Lemma 3.16 guarantees that every irreducible geometric component C of every member of $|D + mK_S|$ is a smooth rational curve. Corollary 3.17 guarantees that S admits an irreducible curve C defined over k of arithmetic genus zero and having non-negative self-intersection.

Finally, we show that if we choose such a C to have minimal degree with respect to some projective embedding, then this self-intersection number is at most two. It remains only to analyze the three cases where C has self-intersection number 0, 1, or 2, which we do in Lemma 3.21.

Putting these steps together, we have proved that if S is a smooth surface of irregularity zero which admits no bicanonical curves and whose Picard group is not generated by its canonical class, then S must be isomorphic to either the projective plane, a quadric surface, or a conic bundle over a geometrically rational curve. This completes the proof of Theorem 3.14. We now proceed to fill in the details.

We begin with the termination of adjunction lemma.

LEMMA 3.15. *Let S be a smooth projective surface of irregularity zero whose bicanonical linear system is empty. Then for any effective divisor D on S, the linear system $|D + nK_S|$ is empty for sufficiently large n.*

PROOF. Given S and D, the conclusion does not depend on the base field, which we may thus assume to be algebraically closed. If $g : S \to T$ is a birational morphism of smooth surfaces, then the image of the linear system $|D + nK_S|$ is contained in the linear system $|g_*D + nK_T|$. By g_*D here we mean the birational transform of D on T, that is, the image of D under g. Hence it is sufficient to prove the lemma for minimal surfaces.

We treat first the case where K_S^2 is non-negative. Recall the Riemann–Roch formula for a divisor $\tilde{D}$ on a smooth surface S:

$$\chi(\mathcal{O}_X(\tilde{D})) - \chi(\mathcal{O}_S) = \tfrac{1}{2}\tilde{D} \cdot (\tilde{D} - K_S),$$

where $\chi(\mathcal{L})$ denotes the alternating sum of the dimensions of the cohomology groups of $\mathcal{L}$. (See, for example, Reid (1997, p. 77) or Hartshorne (1977, V.1.6).) Applied to the divisor $\tilde{D} = -K_S$, the Riemann–Roch formula tells us that

$$h^0(S, \mathcal{O}(-K_S)) + h^2(S, \mathcal{O}(-K_S)) \geq K_S^2 + \chi(\mathcal{O}_S).$$

By Serre duality, $h^2(S, \mathcal{O}(-K_S)) = h^0(S, 2K_S)$, whence for any surface satisfying the vanishing hypothesis of the lemma,

$$h^0(S, \mathcal{O}_S(-K_S)) \geq K_S^2 + \chi(\mathcal{O}_S) \geq 1.$$

This means that $-K_S$ may be taken to be a nonzero effective divisor. But now if the linear system $|D + nK_S|$ is not empty, then some component of some member of $|D|$ contains $n(-K_S)$. But this can not hold for all large n, so the proof is complete in this case.

Now, consider the case where K_S^2 is negative. In this case, $(D + nK_S) \cdot K_S$ is negative for large n. Hence, if $|D + nK_S|$ is non-empty, then there is an irreducible component C of some member of $|D + nK_S|$ such that $C \cdot K_S < 0$. By the adjunction formula (3.3.1)

$$C^2 = 2p_a(C) - 2 - C \cdot K_S \geq -1.$$

If equality holds, then C is a -1-curve, contrary to our assumption that S is a minimal surface. Thus $C^2 \geq 0$ and so C has non-negative intersection with any effective divisor, in particular with $D + nK_S$. However, since $C \cdot K_S$ is negative, it follows that $C \cdot (D + nK_S)$ is negative for large n. Hence, $D + nK_S$ can not be linearly equivalent to any effective divisor, and the lemma is proved. □

In the setting of the above lemma, there is a largest $n = n(D)$ such that $|D + nK_S|$ is not empty. We will focus on this linear system. In order to simplify notation, we assume that $n = 0$, that is, that D is effective but $|D + K_S|$ is empty. We will be able to draw very strong conclusions as long as D is not the zero divisor, and we concentrate on this case for the rest of the section.

LEMMA 3.16. *Let S be a smooth projective surface of irregularity zero and let D be an effective divisor on S such that $|D + K_S|$ is empty. Then*

1. *every irreducible geometric component of D is isomorphic to $\mathbb{P}^1$;*
2. *the arithmetic genus of every reduced subdivisor of D is zero;*
3. *if $D - mK_S$ is ample for some $m \geq 0$, then some irreducible geometric component of D has self-intersection number greater than or equal to -1.*

PROOF. Fix a reduced nontrivial sub-divisor D' of D. Applying Riemann–Roch to $D' + K_S$, we see that

$$h^0(S, \mathcal{O}_S(D' + K_S)) + h^0(S, \mathcal{O}_S(-D')) \geq \frac{(D' + K_S) \cdot D'}{2} + \chi(\mathcal{O}_S).$$

The first term on the left vanishes by assumption and the second is also zero since D' is effective. Since $\chi(S, \mathcal{O}_S)$ is positive, we conclude that

$$(D' + K_S) \cdot D' \leq -2\chi(\mathcal{O}_S) \leq -2. \tag{3.16.4}$$

If D' is connected, then $h^0(\mathcal{O}_{D'}) = 1$, so that the adjunction formula implies that

$$2p_a(D') = 2 + (D' + K_S) \cdot D' \leq 2 - 2 = 0.$$

This establishes (2) for connected D'. The non-connected case follows easily. Furthermore, since a reduced and irreducible curve of arithmetic genus zero over an algebraically closed field is isomorphic to a projective line, (1) follows as well.

To prove (3), assume to the contrary that $D_i^2 \leq -2$ for every component D_i of D. Then since D_i is isomorphic to $\mathbb{P}^1$, the adjunction formula tells us that

$$D_i \cdot K_S = -2 - D_i^2 \geq 0$$

for every D_i. In particular, $D \cdot K_S$ is non-negative. So if $D - mK_S$ is ample, then

$$(D + K_S) \cdot D = ((D - mK_S) + (m+1)K_S) \cdot D \geq (m+1)(K_S \cdot D) \geq 0.$$

But we know from the inequality (3.16.4) that $(D + K_S) \cdot D$ is less than or equal to -2. This contradiction completes the proof of (3). □

COROLLARY 3.17. *Let S be a smooth projective minimal surface of irregularity zero over a perfect field k. Assume that there is an ample divisor H not linearly equivalent to a multiple of K_S. Then S admits a reduced and irreducible curve C of arithmetic genus zero such that*

1. $C \cdot K_S < 0$,
2. $C \cdot C_i \geq 0$ *for every irreducible component* $C_i \subset C_{\bar{k}}$.

PROOF. By the termination of adjunction Lemma 3.15, we may fix m largest possible such that $|H + mK_S|$ is not empty. Let D be a divisor in this linear system. By Lemma 3.16(3), some geometric component, call it C_1, of D has self-intersection at least -1. Considering C_1 as a curve on $S_{\bar{k}}$, let $C := C_1 + \cdots + C_n$ be the sum of all its conjugates. Then C is defined and irreducible over k and $C \cdot K_S = n(-2 - C_1^2) \leq -n$. This establishes (1).

If $(C_1 \cdot C_1) \geq 0$ then $(C \cdot C_1) \geq 0$. Next consider the case when $(C_1 \cdot C_1) = -1$.

If the curves C_i are all disjoint, then C is a -1-curve, contradicting minimality of S. Otherwise, C_1 intersects $(C_2 + \cdots + C_n)$ and so

$$C \cdot C_1 = C_1^2 + C_1 \cdot (C_2 + \cdots + C_n) \geq -1 + 1 = 0,$$

establishing (2).

Finally, to see that C has arithmetic genus zero, note that C is a reduced subdivisor of D defined over k, so we may apply Lemma 3.16. □

LEMMA 3.18. *Let S be a smooth projective surface of irregularity zero over a perfect field k. Let C be a reduced curve of arithmetic genus zero on S such that $C \cdot C_i \geq 0$ for every irreducible component $C_i \subset C$. Then*

1. *the linear system $|C|$ is base point free;*
2. *if C^2 is positive, then the dimension of $|C|$ is $C^2 + 1$ and the general member of $|C|$ over $\bar{k}$ is an irreducible geometrically rational curve.*

PROOF. We first prove (1). Consider the line bundle $L = \mathcal{O}_S(C)|_C$ on the curve C. Our assumptions imply that L has non-negative degree on each component of C. Thus by Exercise 3.4(b), the bundle L is generated by its global sections on C. Now if P is a base point of $|C|$, then P necessarily lies on the curve C. Because L is globally generated, we can choose a section s of L that does not vanish at P. By the long exact sequence of cohomology

$$0 \to H^0(S, \mathcal{O}_S) \to H^0(S, \mathcal{O}_S(C)) \to H^0(C, \mathcal{O}_S(C)|_C) \to H^1(S, \mathcal{O}_S) = 0,$$

the section s must lift to a section of $\mathcal{O}_S(C)$, of course also not vanishing at P. This means that P can not be a base point of $|C|$ after all, and (1) is proved.

To prove (2), first note that because C^2 is positive, the curve C is connected by Exercise 3.19 below. In this case, the long exact sequence above implies that

$$\dim H^0(S, \mathcal{O}_S(C)) = 1 + \dim H^0(C, \mathcal{O}_S(C)|_C) = C^2 + 2,$$

where the last equality follows from Exercise 3.4(c).

To show that the general member of $|C|$ is irreducible, we use the form of Bertini's theorem proved in Exercise 3.20 below. Since $|C|$ is base point free, it has no fixed curves, and because C^2 is positive, it can not map S to a curve. So Bertini's theorem implies that the general member of $|C_{\bar{k}}|$ is irreducible. $\square$

EXERCISE 3.19. Let B be an effective non-connected curve on a smooth surface. Prove that either B has zero intersection number with all of its irreducible components or B has negative intersection number with at least one of its irreducible components.

EXERCISE 3.20 (Bertini theorem). Let M be a mobile linear system on a variety over an algebraically closed field. Show that either

1. the image of the map defined by M is a curve; or
2. the general member of M is irreducible.

LEMMA 3.21. *Let S be a smooth projective minimal surface of irregularity zero over a perfect field k. Assume that the canonical bundle of S does not generate the Picard group. Among all reduced curves C on S such that*

$$H^1(C, \mathcal{O}_C) = 0 \qquad \textit{and} \qquad C \cdot C_i \geq 0$$

for every geometric irreducible component C_i of C, chose one so that the degree of C is minimal with respect to some projective embedding of S. Then C^2 is at most two, and the map given by $|C|$ realizes S in one of the following three ways:

1. *as a conic bundle over a rational curve;*
2. *as isomorphic to* $\mathbb{P}^2$*; or*
3. *as isomorphic to a smooth quadric surface in* $\mathbb{P}^3$.

The three cases above occur when C^2 *is* 0, 1, *or* 2*, respectively.*

PROOF. By assumption there is a divisor A which is not linearly equivalent to a multiple of K_S. Let H be any ample divisor on S. Choose b such that $bH + A$ is also ample. H and $bH + A$ can not both be linearly equivalent to a multiple of K_S, thus by Corollary 3.17 there is a reduced and irreducible curve C of arithmetic genus zero such that $C \cdot K_S < 0$ and $C \cdot C_i \geq 0$ for every irreducible component $C_i \subset C_{\bar{k}}$. We can thus choose one whose degree is minimal with respect to some projective embedding of S.

Let us first assume that $C^2 \leq 2$. By Lemma 3.18, the linear system $|C|$ is base point free, so we need only analyze its image in each of the three cases $C^2 = 0$, 1, or 2.

If $C^2 = 0$, then $|mC|$ maps S onto a curve B with connected fibers for $m \gg 1$. Note also that because $H^1(\mathcal{O}_B)$ injects into $H^1(\mathcal{O}_S) = 0$ and B is geometrically irreducible, the curve B is geometrically rational. This also implies that all geometric fibers are linearly equivalent. Let D be an irreducible fiber, defined over $\bar{k}$. Then $D \cdot K_S < 0$ since $C \cdot K_S = -2 < 0$ and also $D^2 = 0$. By the adjunction formula D is a smooth rational curve and so $S \to B$ is a conic bundle by Exercise 3.13.6.

If $C^2 > 0$, then $\dim |C| = C^2 + 1$ by Lemma 3.18.2.

If $C^2 = 1$, then $\dim |C|$ is two, so we get a surjective morphism $\phi : S \to \mathbb{P}^2$. Since the hyperplane system H of $\mathbb{P}^2$ pulls back to C, we compute that

$$1 = C^2 = \deg \phi \cdot H^2 = \deg \phi.$$

This means that ϕ must be birational. Because S is minimal, the map ϕ is therefore an isomorphism.

If $C^2 = 2$, we get a morphism $\phi : S \to T \subset \mathbb{P}^3$ onto a surface of degree at least 2. Since

$$2 = C^2 = \deg T \cdot \deg \phi \geq 2 \deg \phi,$$

we conclude that ϕ is birational onto a quadric. If T is smooth, then again because S is minimal, we see that ϕ is an isomorphism. The only other possibility is that T is a quadric cone, but this does not happen because of our minimality assumption on the degree of $|C|$. Indeed, if T is a quadric cone, then let D be the linear subsystem of $|C|$ consisting of the pull backs of those hyperplane sections of T that pass through its vertex. In this case, the fiber over the vertex is a one-dimensional component of the base locus of D; removing it, the mobile part of D is a linear system on S of strictly smaller degree satisfying all the requirements, a contradiction.

It remains to prove that $C^2 \leq 2$.

Let $D \in |C|$ be any member defined over k. As in the proof of Corollary 3.17, there is an irreducible and reduced subcurve $D' \subset D$ such that $D' \cdot D' \geq 0$. On the other hand, the minimality assumption on the degree of C implies that hence $D' = D$. Using Exercise 3.4(c), we conclude that

4. Every $D \in |C|$ defined over k is either geometrically irreducible, in which case it is a smooth geometrically rational curve, or it has two geometric irreducible components, each a smooth geometrically rational curve over $\bar{k}$, conjugate over k. In particular, D has at most one singular point.

We prove that $C^2 \leq 2$ first under the following assumption: *There are two distinct k-rational points of S*. Let us call the points P_1 and P_2.

If $C^2 \geq 3$, then $|C|$ has dimension at least four, and so there is a divisor $|C|$ with a singular point at P_1 and passing through P_2. Indeed, consider the exact sequence

$$0 \to \mathcal{O}_X(C) \otimes \mathcal{O}_X(-2P_1 - P_2) \to \mathcal{O}_X(C) \to \frac{\mathcal{O}_X}{\mathfrak{m}_{P_1}^2 \cap \mathfrak{m}_{P_2}};$$

since the dimension of $\frac{\mathcal{O}_X}{\mathfrak{m}_{P_1}^2 \cap \mathfrak{m}_{P_2}}$ over k is four, the corresponding left exact sequence of global sections guarantees that the dimension of $H^0(\mathcal{O}_X(C) \otimes \mathcal{O}_X(-2P_1 - P_2))$ is at least one.

Let C' be a divisor in $|C|$ having a singular point at P_1 and passing through P_2. Since C' is singular at P_1, statement (4) shows that it consists of two irreducible components conjugate over k. One of these passes through P_2, hence so does the other. This implies that C' is also singular at P_2, a contradiction to (4).

Unfortunately, we can not always find two k-rational points on S, see for instance Exercise 1.39. However, we can get around this problem by enlarging the field. This is tricky because after enlarging the field, new -1-curves may appear and so Lemma 3.21 need not apply. There are at least two ways of solving this problem. One is outlined in Proposition 3.22, which states that in fact it is possible to enlarge the field so as to find two rational points on S without introducing any new -1-curves. The other method, which we continue with here, is more geometric.

We begin with the observation that there is a quadratic extension k' of k and a pair of distinct k'-rational points on S that are conjugate over k. Indeed, let $D \in |C|$ be any member. By Exercise 3.4(c), the curve D is isomorphic to a plane conic, and so it always has a pair of conjugate points defined over a quadratic extension.

Now, if $C^2 \geq 5$ then the dimension of $|C|$ is at least six. In this case, there is a divisor $C' \in |C|$ with a singular point at both P_1 and P_2. This, however, contradicts Lemma 3.21(4).

If $C^2 = 3$ or 4, then the dimension of $|C|$ is at least four. In this case, there is a divisor $C_1 \in |C_{k'}|$ with a singular point at P_1 and passing through P_2. (We would expect C_1 to have just two geometric components, but $S_{k'}$ may have -1-curves, so Lemma 3.21(4) need not apply to C_1.)

The curve red C_1 is a tree of smooth rational curves over $\bar{k}$ by Exercise 3.4. Thus, since C_1 has a singular point, it is reducible over $\bar{k}$. Let C_{12} be the reduced subdivisor consisting of those geometric components of C_1 which make up the unique chain of rational curves connecting P_1 and P_2. Let C_{11} be the reduced subdivisor made up of the other components of C_1 through P_1. C_{11} is not empty since C_1 has a singular point at P_1. Note that both C_{11} and C_{12} are defined over k'.

Let C_2 (respectively, C_{22}, C_{21}) be the conjugate of C_1 (respectively, C_{11}, C_{12}). Without loss of generality, we can assume that C_1 and C_2 have no common components. Indeed, if they did, then the sum of these common components determine a divisor D defined over k, and $|C - D|$ would be a linear system satisfying our hypothesis of strictly smaller degree.

Thus all intersections of C_1 and C_2 are proper and we obtain that

$$\begin{aligned} 4 \geq C_1 \cdot C_2 &\geq C_{11} \cdot C_{22} + C_{11} \cdot C_{21} + C_{12} \cdot C_{22} + C_{12} \cdot C_{21} \\ &\geq C_{11} \cdot C_{22} + \operatorname{mult}_{P_1} C_{11} + \operatorname{mult}_{P_2} C_{22} + 2, \end{aligned}$$

where $C_{12} \cdot C_{21} \geq 2$ since they have at least two intersection points, namely P_1 and P_2. Thus $\operatorname{mult}_{P_1} C_{11} = \operatorname{mult}_{P_2} C_{22} = 1$ and $C_{11} \cdot C_{22} = 0$. The first of these implies that C_{11} is an irreducible curve and the second shows that $C_{11} + C_{22}$ is defined over k and has two disjoint components over k'. Thus C_{11}^2 is negative by Exercise 3.19. Therefore $C_{11} + C_{22}$ is a -1-curve, contrary to our assumptions. □

The following proposition presents a different way to complete the proof of Lemma 3.21 in the case where S does not admit two distinct k-rational points.

PROPOSITION 3.22. *Let S be a smooth projective geometrically irreducible surface over a field K. Then, for every n there is a field extension L of K such that there are at least n L-points on S and every -1-curve on S_L is already defined over K.*

PROOF. Setting $L_1 := K(S)$, the generic point of S gives an L_1-point of S. Iterating the procedure we get more and more points. By Exercise 3.32, we can conclude that no new -1-curves appear on S_L if the field K is algebraically closed in L. Thus it suffices to show that K is algebraically closed in $K(S)$.

To this end, suppose to the contrary that L is an algebraic extension of K contained in $K(S)$. Then the structure map $S \to \operatorname{Spec} K$ factors through $S \to \operatorname{Spec} L$. Thus $S_{\bar{K}}$ maps onto $\operatorname{Spec} L \times_{\operatorname{Spec} K} \operatorname{Spec} \bar{K}$ which is the union of $|L/K|$ closed points if L/K is separable. This is impossible since $S_{\bar{K}}$ is irreducible. (In positive characteristic it may happen that $\operatorname{Spec} L \times_{\operatorname{Spec} K} \operatorname{Spec} \bar{K}$ has only one closed point, in which case this scheme has nilpotents. Now we use that $S_{\bar{K}} \to \operatorname{Spec} L \times_{\operatorname{Spec} K} \operatorname{Spec} \bar{K}$ is faithfully flat to get a contradiction to the smoothness of S.) □

EXERCISE 3.23. Find an example of a smooth surface over K such that S_K has no -1-curves but such that $S_{K'}$ has -1-curves for every nontrivial algebraic field extension $K' \supset K$.

3.4 Field of definition of a subvariety

Fix a field extension K of a field k. In this section, we treat in detail the following question: *Given a variety defined over K, how can we tell whether it is actually defined over the smaller field k?*

Many other questions can be reduced to this. For instance, the rationality of a variety X is equivalent to the existence of a subvariety $\Gamma \subset X \times \mathbb{P}^n$ with certain properties (making Γ into the graph of a birational map). As we pointed out early in Chapter 1 (see Example 1.3), the variety X can be rational over K but not over k; this corresponds to Γ being defined over K but not over k.

DEFINITION 3.24. Let K/k be a field extension and let X be a variety defined over K. We say that a k-variety Y is a *k-form of* X if there exists an isomorphism of varieties over K

$$X \cong Y \times_{\operatorname{Spec} k} \operatorname{Spec} K.$$

Existence and uniqueness of k-forms are quite interesting and subtle questions in general. Given a variety X_K, a k-form need not exist, and even if it does, it need not be unique. For instance, any smooth plane conic over $\mathbb{Q}$ is a $\mathbb{Q}$-form of $\mathbb{P}^1_{\mathbb{C}}$. On the other hand, if K is a Galois extension of k and X has no automorphisms then k-forms are unique (but they need not exist). (See Serre (1979, Ch. X) for an introduction to such questions and to Galois cohomology.)

One can define k-forms for just about any object in algebraic geometry. For illustration, we define k-forms of sheaves and subschemes.

Let X be a k-scheme, $\mathcal{F}$ a coherent sheaf on X_K and $\mathcal{G}$ a coherent sheaf on X_k. We say that $\mathcal{G}$ is a *k-form* of $\mathcal{F}$ if there is an isomorphism of $\mathcal{O}_{X_K}$-modules

$$\pi^*\mathcal{G} \cong \mathcal{F}$$

where π is the natural projection $X_K = X_k \times_{\operatorname{Spec} k} \operatorname{Spec} K \xrightarrow{\pi} X_k$.

If $\mathcal{F}$ and $\mathcal{G}$ are ideal sheaves, then there is a stronger notion. Namely, one can insist that $\pi^*\mathcal{G}$ is actually equal to $\mathcal{F}$, not merely isomorphic as sheaves. Similarly, one can insist on equality in Definition 3.24, in the case that the subschemes in question are contained in some fixed ambient scheme. This actually simplifies the question considerably and it deserves new definitions.

DEFINITION 3.25. Let X be a k-scheme and let W be a subscheme of $X_K = X \times_{\operatorname{Spec} k} \operatorname{Spec} K$. We say that W is *defined over k* or that *k is a field of definition of W* if there is a subscheme V of X defined over k such that

$$W = V \times_{\operatorname{Spec} k} \operatorname{Spec} K,$$

where we view both sides as subschemes of X_K. We prove below in Theorem 3.26 that V is unique (if it exists).

The reader should keep in mind that there is a subtle but significant difference between the two notions defined in Definitions 3.24 and 3.25. For instance, consider the variety $X = \mathbb{P}^1$ defined over the field $k = \mathbb{Q}$. Let K be the field extension $\mathbb{Q}(\sqrt{2})$. The subvariety W consisting of the two points $(0, 1)$ and $(\sqrt{2}, 1)$ is defined over K but not over k. On the other hand, any set of two points defined over k is a k-form of the scheme W.

The following basic result due to Weil asserts that a subscheme has a minimal field of definition.

THEOREM 3.26. (Weil, 1962, I.7 Lemma 2) *Let K/k be a field extension. Let X_k be a scheme defined over k and let W be a closed subscheme of X_K. Then*

1. *there is at most one subscheme $V \subset X_k$ such that $W = V_K$;*
2. *there is a unique smallest field L between k and K such that W is defined over L.*

PROOF. Assume first that $X_k = \operatorname{Spec} R$ is affine. Then W corresponds to an ideal $I \subset R \otimes_k K$ and V to an ideal $J \subset R$. Since $J \otimes_k K = I$, it is clear that $J = (J \otimes_k K) \cap R = I \cap R$. This shows that J is unique, proving (1).

In order to prove (2), we need to show that there is a unique smallest field L such that I has a system of generators in $R \otimes_k L$. It turns out to be easier to think of this as a vector space question, and forget the multiplicative structure of R. Thus we are looking for a smallest field L such that I has a basis in $R \otimes_k L$.

Let $\{M_\lambda : \lambda \in \Lambda\}$ be a k-basis of R. Fix a well-ordering of the indexing set Λ. Set

$$\Lambda' := \left\{ \lambda \in \Lambda \mid M_\lambda \in I + \sum_{\mu<\lambda} K \cdot M_\mu \right\} \quad \text{and} \quad \Lambda'' := \Lambda \setminus \Lambda'.$$

Then the vector space spanned by $\{M_\lambda : \lambda \in \Lambda''\}$ intersects I only at the origin, and for every $\lambda \in \Lambda'$ there is a linear relation

$$M_\lambda = i_\lambda + \sum_{\mu<\lambda} c_{\lambda,\mu} M_\mu, \quad \text{for some } i_\lambda \in I,\ c_{\lambda,\mu} \in K. \qquad (3.26.3_\lambda)$$

If the sum involves a nonzero term $c_{\lambda,\mu}M_\mu$ for some $\mu \in \Lambda'$ then we can eliminate this term by substitution using $(3.26.3_\mu)$. The sequence of back substitutions eventually stops since the index set is well ordered. In the end, we get equations

$$M_\lambda = i^*_\lambda + \sum_{\mu<\lambda,\mu\in\Lambda''} c^*_{\lambda,\mu} M_\mu \quad \text{where } i^*_\lambda \in I,\ c^*_{\lambda,\mu} \in K. \qquad (3.26.4_\lambda)$$

Note that $i^*_\lambda \in I$ and $c^*_{\lambda,\mu} \in K$ are unique since otherwise subtracting them would give a linear relation involving elements of I and $\{M_\mu : \mu \in \Lambda''\}$, but as we remarked above, these vector spaces have trivial intersection.

The set

$$B(I) := \left\{ M_\lambda - \sum_{\mu<\lambda,\mu\in\Lambda''} c^*_{\lambda,\mu} M_\mu \mid \lambda \in \Lambda' \right\}$$

is contained in I, and, together with $\{M_\lambda : \lambda \in \Lambda''\}$ it spans $R \otimes_k K$ over K. Thus $B(I)$ is a K-basis of I. In particular, $L = k(\{c^*_{\lambda,\mu} : \lambda \in \Lambda', \mu \in \Lambda''\})$ is a field of definition of I.

We claim that L is the unique smallest subfield of K over which I is defined. Assume that I is definable over a field L' by an ideal J'. Then J' and $\{M_\lambda : \lambda \in \Lambda''\}$ span $R \otimes_k L'$. Hence there are linear relations with coefficients in L'

$$M_\lambda = j_\lambda + \sum_{\mu<\lambda,\mu\in\Lambda''} d_{\lambda,\mu} M_\mu \quad \text{where } j_\lambda \in J',\ d_{\lambda,\mu} \in L'.$$

By uniqueness of the equations $(3.26.4_\lambda)$, we conclude that $c^*_{\lambda,\mu} = d_{\lambda,\mu}$, hence $L \subset L'$. Therefore L is the unique smallest field of definition of I.

For an arbitrary scheme X, let $\cup U_i = X$ be an affine cover of X. Each $(W \cap U_i)$ has a smallest field of definition L_i. Let $V_i \subset U_i$ be the corresponding subschemes and $L \subset K$ the composite of the L_i. $V_i \cap U_j$ and $V_j \cap U_i$ are forms of $W \cap (U_i \cap U_j)$ over L, hence they agree by uniqueness in the affine case. Thus the V_i patch together to a subscheme $V \subset X_L$. □

EXERCISE 3.27. Consider the plane conic C over $\mathbb{Q}$ defined by the homogeneous equation $3x^2 + 5y^2 = z^2$. On $C_{\bar{\mathbb{Q}}}$, consider the line bundle $L := \mathcal{O}(1)$. Prove that L is defined over a field K if and only if C has an K-point. Thus L is defined over $\mathbb{Q}(\sqrt{3})$ and over $\mathbb{Q}(\sqrt{5})$ but it is not defined over their intersection, $\mathbb{Q}$. Thus there need not be a minimal field of definition for a line bundle over a variety.

The example of Exercise 3.27 also shows, by combining with Proposition 1.7, that there is no minimal field over which the conic C is rational.

3.28. GALOIS ACTION ON VARIETIES. Let K be a (possibly transcendental) algebraically closed extension of a fixed ground field k (for instance, the extension $\mathbb{C}$ of $\mathbb{Q}$.) Let $G = \mathrm{Aut}(K/k)$ denote the group of k-automorphisms of K. If K is the algebraic closure of k, then G is essentially the absolute Galois group.[1]

For any n, the group G acts naturally on the K-points of $\mathbb{P}^n$ by sending a point with coordinates $(x_0 : \cdots : x_n)$ to the point $(g(x_0) : \cdots : g(x_n))$.

If $f = \sum a_I x^I \in K[x_0, \ldots, x_n]$ is homogeneous then

$$f(g(x_0) : \cdots : g(x_n))) = g(f^g(x_0 : \cdots : x_n))$$

where $f^g = \sum g^{-1}(a_I)x^I$. Thus if X is a subvariety of $\mathbb{P}^n_K$ defined by the polynomials

$$f_1, \ldots, f_m,$$

then $g(X)$ is the subvariety of $\mathbb{P}^n_K$ defined by

$$f_1^g, \ldots, f_m^g.$$

If X is defined by equations in $k[x_0, \ldots, x_n]$ then we get a G-action on $X(K)$. It is easy to see that this action depends only on X and not on the embedding $X \hookrightarrow \mathbb{P}^n$. It is also rather obvious that in the case where X is a surface, this G-action respects the intersection product for curves.

CAUTION 3.29. The Galois action on varieties seems quite simple, but it can be quite confusing.

The main reason is that it is tricky to interpret the sense in which these maps are or are not scheme automorphisms, although they do act like automorphisms from many points of view.

Indeed, G gives only the identity action on the k-points of the k-scheme $\mathbb{A}^n_k = \mathrm{Spec}_k\, k[x_1, \ldots, x_n]$, since $(x_1, \ldots, x_n)$ and $(g(x_1), \ldots, g(x_n))$ always

[1] Usually the absolute Galois group is defined to be the inverse limit of the Galois groups of all finite subextensions, but this distinction is not important in our current discussion.

correspond to the same closed point. We do get a good action on the K-points of the K-scheme $\mathbb{A}^n_K = \operatorname{Spec}_K K[x_1, \ldots, x_n]$. This action is k-linear, but not K-linear, so it is not a K-scheme action.

On the other hand, G does give an honest k-scheme action on the k-scheme $W_k = \operatorname{Spec}_k K[x_1, \ldots, x_n]$, but this scheme is not affine n-space over k. Indeed, as a k-scheme, W_k is $\operatorname{Spec}_k K \times_k \operatorname{Spec}_k k[x_1, \ldots, x_n]$, so it can be thought of as a disjoint union of $|G| = [K : k]$ copies of $\mathbb{A}^n_k$.

Fortunately, the situation is fairly clear as long as we stick to actions on points.

Corollary 3.30. *Let K/k be a field extension with K algebraically closed. Let H be a subgroup of the full group of k-automorphisms of K, and let L be the subfield of K fixed by H. Let X be a variety defined over k, and let W be a subvariety of X_K. Then W is defined over L if and only if W is invariant under H.*

Proof. It is enough to consider the affine case. Let I be the ideal of W. Using the notation of the proof of Theorem 3.26, the equations (3.26.4$_\lambda$) transform under $g \in H$ into

$$M_\lambda = g^{-1}(i^*_\lambda) + \sum_{\mu<\lambda;\mu\in\Lambda''} g^{-1}(c^*_{\lambda,\mu})M_\mu. \qquad (3.30.1_\lambda)$$

If I is G-invariant, then these equations satisfy the same assumptions as (3.26.4$_\lambda$), hence by uniqueness we see that $g^{-1}(c^*_{\lambda,\mu}) = c^*_{\lambda,\mu}$. This implies that $c^*_{\lambda,\mu} \in L$, so W is defined over L. □

Exercise 3.31. Let K/k be a field extension with K algebraically closed. Let X be a k-variety and $W \subset X_K$ a subvariety. Then W is defined over an algebraic extension of k if and only if the orbit of W under the action of the automorphism group $\operatorname{Aut}(K/k)$ is finite.

Exercise 3.32. Let S_k be a smooth projective surface over a field k. Let K be a field extension of k such that k is algebraically closed in K. Show that every irreducible curve on S_K with negative self-intersection is defined over k.

We have seen in this section how the properties of a variety change under a field extension. In practice, we frequently work inside a fixed algebraically closed field, say $\mathbb{C}$, and it is useful to know that further extensions of $\mathbb{C}$ do not produce anything new. In fact, properties of algebraic varieties rarely change when we pass from one algebraically closed field to another. Assertions of this type are sometimes called *Lefschetz principles*. We illustrate this principle in a simple case.

PROPOSITION 3.33. *Let X be a variety defined over an algebraically closed field K. Let L be any field extension of K. Then X is rational over K if and only if it is rational over L.*

PROOF. If X is rational over K, then it is rational over any extension field, including L.

Conversely, assume that X is rational over L and let $\phi : \mathbb{P}^n \dashrightarrow X$ be a birational map. The coordinate functions of ϕ can be expressed using only finitely many elements of L, thus ϕ is defined over a finitely generated subextension $F \subset L$. We can realize F as the function field of some K-variety Y, and so we can think of ϕ as a map $\phi : \mathbb{P}^n \times Y \dashrightarrow X$. Taking its product with the second projection map we obtain a birational map

$$\Phi : \mathbb{P}^n \times Y \dashrightarrow X \times Y,$$

which commutes with projection to Y. Thus the induced map of K-varieties

$$\Phi_y : \mathbb{P}^n \times \{y\} \dashrightarrow X \times \{y\}$$

is birational for all points y in a suitable open subset of Y. Finally, because K is algebraically closed, the closed points of the K-variety Y are all defined over K, so the map Φ_y defines a birational map $\mathbb{P}^n \dashrightarrow X$ over K. □

Finally, we settle a potentially troubling question about computing cohomologies and base fields that we have been avoiding for too long. Fortunately, the answer is that there is nothing to worry about.

EXERCISE 3.34. Let X be a scheme defined over a field k and let F be a coherent sheaf on X (thus also defined over k). Let K be an extension field of k. Prove that $H^i(X_K, F_K) \cong H^i(X, F) \otimes_k K$.

3.5 Del Pezzo surfaces

An important class of geometrically rational surfaces are the *Del Pezzo* surfaces. The theory of Del Pezzo surfaces and their higher dimensional counterparts is one of the cornerstones of the modern point of view on the birational classification problem.

DEFINITION 3.35. A Del Pezzo surface is a smooth projective geometrically irreducible surface whose anti-canonical bundle is ample.

Slightly singular varieties with ample anti-canonical bundle are also of interest. These singular Del Pezzo surfaces are treated in the exercises.

It is easy to see that Del Pezzo surfaces are geometrically rational. Indeed, according to Castelnuovo's criterion (Theorem 3.2), a smooth surface S is geometrically rational if

$$H^0(S, 2K_S) \quad \text{and} \quad H^1(S, \mathcal{O}_S)$$

both vanish. When S is a Del Pezzo surface, the former vanishes because $-2K_S$ is ample, so certainly $2K_S$ is not effective. In characteristic zero, the latter vanishes by the Kodaira vanishing theorem, since the trivial divisor can be written as $K_S + H$ where $H = -K_S$ is ample. Even in characteristic p, this vanishing holds (see, for instance, Kollár (1996, III.3)) and so every Del Pezzo surface is geometrically rational.

The *degree* of a Del Pezzo surface is the self-intersection number of its canonical class:

$$\deg S = K_S^2.$$

Note that the degree of a Del Pezzo surface is a positive integer.

The main result of this section is the following classification theorem for Del Pezzo surfaces over an arbitrary field. Of course, whether or not a surface is Del Pezzo does not depend on the field of definition, but it is still of interest to understand the different isomorphism types of Del Pezzo surfaces over a given field.

THEOREM 3.36. 1. *The degree of a Del Pezzo surface is at most nine.*
2. *If a Del Pezzo surface has degree greater than two, then its anti-canonical linear system is very ample.*
3. *A degree four Del Pezzo surface is isomorphic to a complete intersection of two quadrics in* $\mathbb{P}^4$.
4. *A degree three Del Pezzo surface is isomorphic to a cubic surface in* $\mathbb{P}^3$.
5. *A degree two Del Pezzo surface is isomorphic to a hypersurface of degree four in the weighted projective space* $\mathbb{P}(2, 1, 1, 1)$. *(Weighted projective spaces are reviewed in 3.48.)*
6. *A degree one Del Pezzo surface is isomorphic to a degree six hypersurface in the weighted projective space* $\mathbb{P}(3, 2, 1, 1)$.
7. *Any smooth surface as in (3), (4), (5) or (6) is a Del Pezzo surface of the expected degree.*

Our proof of Theorem 3.36 is valid only in characteristic zero because we use the Kodaira vanishing to prove Lemma 3.38. The theorem, however, remains true for any field; see, for instance, Kollár (1996, III.3). For related works see Demazure et al, (1980) and Manin (1986).

REMARK 3.37. Theorem 3.36 completely classifies Del Pezzo surfaces of degree at most four over an arbitrary field. It is possible to give a complete description of the remaining Del Pezzo surfaces as well. See, for instance, Manin (1986). From the arithmetic and geometric points of view the higher degree Del Pezzo surfaces are simpler than the low degree ones. On the other hand, there does not seem to be a unified description of them.

We begin with some easy numerical calculations for the pluricanonical series on a Del Pezzo surface.

LEMMA 3.38. *Let X be a Del Pezzo surface over a field of characteristic zero. Then X has irregularity zero and the dimension of the linear series $|-mK_X|$ is*

$$\frac{m(m+1)}{2}\deg(X)$$

for all $m \geq 0$.

We prove this only in characteristic zero. See Kollár (1996, III.3) for the prime characteristic case.

PROOF. First note that the higher cohomologies of $\mathcal{O}_X(-mK_X)$ all vanish, for $m \geq 0$. Indeed, by Serre duality $h^2(X, \mathcal{O}(-mK_X)) = h^0(X, \mathcal{O}((1+m)K_X))$, and the latter is zero since $-K_X$ is ample. For H^1, we dualize and use Kodaira vanishing: $h^1(X, \mathcal{O}(-mK_X)) = h^1(X, \mathcal{O}(K_X + (1+m)(-K_X))) = 0$.

Also, by the Riemann–Roch Formula, we have that

$$\chi(X, \mathcal{O}_X(-mK_X)) = \frac{(-mK_X)\cdot(-mK_X - K_X)}{2} + \chi(X, \mathcal{O}_X)$$

Thus, by the vanishing just proven, we conclude that the dimension of $|-mK_X|$ is

$$\frac{m(m+1)}{2}K_X^2.$$

The proof is complete. □

The members of the anti-canonical linear system $|-K_X|$ on a surface X are called anti-canonical curves. The following proposition describing anti-canonical curves on a Del Pezzo surface is the key step in the proof of Theorem 3.36.

PROPOSITION 3.39. 1. *Every proper subcurve of an anti-canonical curve on a Del Pezzo surface has irregularity zero.*
2. *A general anti-canonical curve on a Del Pezzo surface is an irreducible and reduced curve of arithmetic genus one.*

PROOF. Let C' be a proper subcurve of any anti-canonical curve C on a Del Pezzo surface X. Then $|C' + K_X|$ is empty. This implies that the arithmetic genus of C' is zero by Lemma 3.16. This establishes (1).

For (2), consider any anti-canonical curve C in $|-K_X|$. The adjunction formula implies that

$$2p_a(C) - 2 = C \cdot (C + K_X)$$

where $p_a(C)$ is the arithmetic genus, $\dim H^1(C, \mathcal{O}_C)$ of C. Since $C \sim -K_X$, the right hand since is zero, whence the arithmetic genus is one. To complete the proof of (2), we must show that a general member of $|-K_X|$ is reduced and irreducible.

To this end, we use Bertini's theorem, Exercise 3.20. Write $|-K_X| = |M| + B$ where $|M|$ is mobile and B is fixed.

If $|M|$ maps onto a surface, then a general curve $C \in |M|$ is reduced and irreducible by Bertini's theorem. So we are done if $B = 0$. But if B is not 0, then C is a proper subcurve of an anti-canonical curve, and hence has arithmetic genus zero by (1). By the adjunction formula, therefore,

$$C^2 + 2 = -K_X \cdot C \leq K_X^2 = \deg X$$

By Lemma 3.18, the dimension of $|C|$ is at most $\deg X + 1$. thus

$$\dim |-K_X| = \dim |C| < \deg(X)$$

contradicting Lemma 3.38.

If $|M|$ maps onto a curve, then a general curve $\sum_{i=1}^{s} C_i \in |M|$ may be reducible; in this case, the dimension of $|M|$ is at most the number of components of a general member. On the other hand, by adjunction $-K_X \cdot C_i = C_i^2 + 2 \geq 2$. Thus

$$\dim |-K_X| \leq s \leq \frac{1}{2}\left(-K_X \cdot \sum C_i\right) \leq \frac{1}{2}K_X^2 = \frac{1}{2}\deg(X),$$

which again contradicts Lemma 3.38. □

PROOF OF THEOREM 3.36(1). Fix a Del Pezzo surface X. To show that the degree of X is at most nine, we are free to assume that the ground field is algebraically closed. If the degree of X is at least ten, then by Lemma 3.38 it follows that the dimension of $|-K_X|$ is at least ten as well. Thus, arguing similarly as in the proof of Lemma 3.21, we see that there is an anti-canonical curve with a quadruple point at P.

Let C be an anti-canonical curve with a point of multiplicity four at P. If C is reduced and irreducible, then its arithmetic genus must be one, and we contradict Exercise 3.40 below. Thus C is not reduced and irreducible. Let $C' := \sum a_i C_i$ be the subcurve consisting of those components passing through P. We know that each of the components C_i is a smooth rational curve; see Proposition 3.39.

Assume for now that each component C_i has non-negative self-intersection; we justify this assumption below. In this case, the linear system $|C_i|$ is base point free for each i by Lemma 3.18.

Consider one component C_i of C'. If $C_i^2 = 0$, then the linear system $|C_i|$ maps X to a curve B. By Exercise 3.13(6), then X is a conic bundle over B. But in this case, Exercise 3.13(3) implies that the degree of X is at most eight, a contradiction.

Thus each curve C_i has positive self-intersection. Thus the linear systems $|C_i|$ for all i have dimension at least two, which means that the corresponding linear subsystems of members passing through P has dimension at least one. So by replacing $a_i C_i$ by a linearly equivalent reduced divisor, we may assume that C' is reduced. This contradicts Exercise 3.40 below.

Finally, we justify our assumption that P can be chosen so that the components C_i all have non-negative self-intersection. This uses a basic finiteness result of algebraic geometry. Fix a projective embedding of X. As explained in Shafarevich (1994, VI.4), for any d, the curves of degree at most d and genus g form finitely many connected algebraic families on X. Since intersection numbers are unchanged in connected algebraic families, a curve with negative self-intersection is a connected family by itself. Thus, on any surface S, there are only finitely many curves of degree at most d and with negative self-intersection. Since the canonical curves all have the same fixed degree, say d, with respect to the chosen projective embedding, the components C are all degree at most d and so only finitely many of them can have negative self-intersection. Now we choose P so that it does not lie on any curve of degree at most d having negative self intersection. □

Exercise 3.40. If a reduced curve on a smooth surface has a point of multiplicity m, then its arithmetic genus is at least $\frac{(m-1)(m-2)}{2}$.

Proof of Theorem 3.36(3) and (4). Let X be a Del Pezzo surface. We show that $|-K_X|$ maps X isomorphically onto a cubic surface if $K_X^2 = 3$ and onto a complete intersection of two quadrics if $K_X^2 = 4$. These claims do not depend on the base field, which we may assume to be algebraically closed. Let $C \in |-K_X|$ be a general anti-canonical curve. By Proposition 3.39, C is a reduced and irreducible curve of arithmetic genus one. Since $\mathcal{O}_X(C)|_C$ has degree 3 or 4, it is generated by global sections by Exercise 3.45 below. Since $H^1(X, \mathcal{O}_X) = 0$, the sections of $\mathcal{O}_X(C)|_C$ lift to sections of $\mathcal{O}_X(C)$, and we conclude that $\mathcal{O}_X(-K_X) = \mathcal{O}_X(C)$ is generated by global sections.

Let $\phi : X \to \mathbb{P}^r$ be the morphism given by the anti-canonical linear system, and let H be a general hyperplane in $\mathbb{P}^r$. Then $C := \pi^* H$ is an elliptic curve and $\phi|_C$ is the map given by the very ample line bundle $\mathcal{O}_X(C)|_C$. Thus we see that ϕ is an embedding, except possibly at finitely many points.

If $K_X^2 = 3$, then $\phi(X) \subset \mathbb{P}^3$ is a cubic surface.

If $K_X^2 = 4$, then by Lemma 3.38, we see that the dimension of $H^0(X, \mathcal{O}_X(-2K_X))$ is thirteen. On the other hand, since $H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2))$ has dimension fifteen, there are at least two quadrics Q_1, Q_2 containing $\phi(X)$. Because the intersection of these quadrics is a surface of degree four, it follows that $\phi(X)$ must be equal to this intersection.

There are many ways to see that in fact ϕ is an isomorphism. Conceptually the best is to note that ϕ is finite and birational, hence X is the normalization of the image $\phi(X)$. Thus it is enough to prove that $\phi(X)$ is normal. This is a rather general fact; see Exercise 3.41 below.

In our case there is a simple alternate proof. Set $Y := \phi(X)$. We have an exact sequence

$$0 \to \mathcal{O}_Y \to \phi_*\mathcal{O}_X \to Q \to 0$$

where Q is supported at finitely many points. Thus

$$\chi(\mathcal{O}_X) = \chi(\mathcal{O}_Y) + \text{length}(Q).$$

We know that $\chi(\mathcal{O}_X) = 1$ and we can easily compute that $\chi(\mathcal{O}_Y) = 1$ in the above two cases. Thus $Q = 0$ which means that ϕ is an isomorphism. Parts (3) and (4) of Theorem 3.36 are proved. □

EXERCISE 3.41. Let $Y \subset \mathbb{P}^n$ be a complete intersection variety whose singular set has codimension at least two. Prove that Y is normal.

The description of Del Pezzo surfaces of degree one or two is similar in principle, but there are more details to be worked out. We present them as a series of exercises. We start with the classical approach and then give the modern version involving weighted projective spaces.

EXERCISE 3.42. Let X be a Del Pezzo surface over a field k.

1. If the degree of X is two, then the anti-canonical system is base point free and it represents X as a double cover of the projective plane.
2. If the degree of X is one, then the bi-anticanonical system is base point free and it represents X as a double cover of a quadric cone in $\mathbb{P}^3$.

We get a more invariant description if we look at all the pluricanonical linear systems $|-mK_X|$ simultaneously. This amounts to considering the whole anti-canonical ring of X.

DEFINITION 3.43. Let D be a divisor on a projective scheme X. The section ring of D is the graded ring

$$R(X, D) = \bigoplus_{j\geq 0} H^0(X, \mathcal{O}(jD)),$$

with multiplication induced by the usual multiplication of sections. The section ring of the canonical divisor is called the *canonical ring*, and of the anti-canonical divisor the *anti-canonical ring*.

In the sequel, we need to know when a ring $R(X, D)$ is generated by its elements of a given degree. The following exercise gives an inductive approach to such questions.

EXERCISE 3.44. Let D be an effective Cartier divisor on a scheme X. Assume that

1. The section ring $R(D, \mathcal{O}_X(D)|_D)$ is generated by its elements of degree at most r, and
2. The cohomology groups $H^1(X, \mathcal{O}(jD))$ vanish for all $j > 0$.

Show that the ring $R(X, D)$ is generated by its elements of degree at most r.

EXERCISE 3.45. Let C be an irreducible and reduced curve of arithmetic genus one and let D be a Cartier divisor on C. Show that

1. The section ring $R(C, D)$ is generated by elements of degree one if the degree of D is three or more.
2. The section ring $R(C, D)$ is generated by elements of degrees two and less if the degree of D is two.
3. The section ring $R(C, D)$ is generated by elements of degrees three and less if the degree of D is one.
4. The linear system $|D|$ is base point free if the degree of D is greater than one.
5. The linear system $|D|$ is very ample if the degree of D is greater than two.

EXERCISE 3.46. Prove the following statements.

1. The anti-canonical ring of a Del Pezzo surface of degree at least three is generated by its degree one elements; furthermore, the anti-canonical system is very ample.

2. The anti-canonical ring of a Del Pezzo surface of degree two is generated by its elements of degree at most two; furthermore, the anti-canonical system is base point free.
3. The anti-canonical ring of a Del Pezzo surface of degree one is generated by its elements of degree at most three.

COROLLARY 3.47. *A Del Pezzo surface of degree three or more is is isomorphic to a surface of degree d in $\mathbb{P}^d$.*

Del Pezzo surfaces of degree one or two appear most naturally as hypersurfaces in weighted projective spaces. These are introduced next.

3.48. WEIGHTED PROJECTIVE SPACES. A *weighted homogeneous polynomial* of degree d with *weights* $a_i \in \mathbb{N}$ is a polynomial $f(x_0, \dots, x_n)$ such that $f(y_0^{a_0}, \dots, y_n^{a_n})$ is homogeneous of degree d in the variables $y_0, \dots, y_n$. For example

$$x_0 + x_1^2 + x_1\left(x_2^2 + x_3x_4\right) + x_2^4 + x_2x_3x_4^2$$

is a weighted homogeneous polynomial with weights (4, 2, 1, 1, 1).

Let k be a field. Fix natural numbers $a_0, \dots, a_n$, and assume that any n of the weights are relatively prime. Consider the graded ring $k[x_0, \dots, x_n]$, where the variable x_i has been assigned degree a_i. The degree d graded piece of this ring is exactly the k-vector space of weighted homogeneous polynomial of degree d with weights $a_0, \dots, a_n$.

The *weighted projective space* $\mathbb{P}_k(a_0, \dots, a_n)$ of dimension n with weights $a_0, \dots, a_n$ is the projective scheme given by this graded ring. That is,

$$\mathbb{P}_k(a_0, \dots, a_n) := \operatorname{Proj} k[x_0, \dots, x_n]$$

where the variable x_i has weight a_i. Of course, the ordinary projective space $\mathbb{P}^n$ is recovered as the weighted projective space $\mathbb{P}(1, \dots, 1)$.

How do we map to a weighted projective space? Remember that a map $X \dashrightarrow \mathbb{P}^n$ to ordinary projective space is given by a line bundle L on X together with $n+1$ of its global sections, which give us the coordinates. In a weighted projective space different coordinates have different weights, hence they should correspond to sections of different powers of L. Indeed, a collection of sections $s_i \in H^0(X, L^{a_i})$ determines a map

$$\begin{aligned} X &\dashrightarrow \mathbb{P}_k(a_0, \dots, a_n) \\ P &\mapsto (s_0(P) : s_1(P) : \dots : s_n(P)). \end{aligned}$$

It is not hard to see that every map to weighted projective space is given this way, at least if we restrict our attention to the smooth locus of X and of $\mathbb{P}_k(a_0, \dots, a_n)$.

(There are some problems in general since $\mathcal{O}(1)$ may not be locally free, and so its pull back to X may behave strangely.)

Caution 3.49. A point in $\mathbb{P}(a_0, \dots, a_n)$ can be represented by projective coordinates $(x_0 : \cdots : x_n)$, but the equivalence relation is

$$(x_0 : \cdots : x_n) \sim (\lambda^{a_0} x_0 : \cdots : \lambda^{a_n} x_n).$$

Thus, for instance, in $\mathbb{P}(2, 1, 1)$, we get that $(1 : -1 : -1) = (1 : 1 : 1)$ by taking $\lambda = -1$. These coincidences are easy to overlook.

3.50. Affine charts for weighted projective spaces. In order to get a feeling for the weighted projective spaces $\mathbb{P}(a_0, \dots, a_n)$, we work out the "standard" affine charts on it. To avoid some complications, we always assume that any n of the a_i are relatively prime.

By definition, $\mathbb{P}(a_0, \dots, a_n) = \operatorname{Proj} k[x_0, \dots, x_n]$ where x_i has degree a_i. Let $U_i \subset \mathbb{P}(a_0, \dots, a_n)$ be the set where $x_i \neq 0$. We can then write

$$U_i = \operatorname{Spec} k\big[x_0, \dots, x_n, x_i^{-1}\big]_{(0)}$$

where the subscript (0) denotes the subring of degree 0 elements. In general this ring is quite complicated, but below we write it out explicitly in some nice cases.

If $a_i = 1$, then

$$k\big[x_0, \dots, x_n, x_i^{-1}\big]_{(0)} = k\big[x_0 x_i^{-a_0}, \dots, x_n x_i^{-a_n}\big],$$

where we omit $x_i/x_i = 1$ from the list. In this case U_i is smooth, just like for the standard $\mathbb{P}^n$.

If $a_i = 2$, then the corresponding ring is

$$k\big[x_j x_i^{-a_j/2} : a_j \text{ even}, \quad x_s x_t x_i^{-(a_s+a_t)/2} : a_s, a_t \text{ odd}\big].$$

From this we see that the origin is always a singular point of U_i whenever $a_i = 2$. More generally, it turns out that the origin is a singular point of U_i whenever $a_i > 1$.

Conceptually, the best way to think about the charts U_i is to work with *orbifold coordinates*.

By looking at the example of $a_i = 2$, it is clear that we would like to have generators $x_j / x_i^{a_j/2}$, except that these do not exist if a_j is odd. Nonetheless, this idea gives very useful coordinates the following way.

Let us consider a ring

$$k[u_0, \dots, \hat{u}_i, \dots, u_n]$$

where we try to pretend that $u_j = x_j / x_i^{a_j/a_i}$. If $\prod u_j^{m_j}$ is a monomial in the u_js, then our pretending identifies it with

$$x_i^{-\frac{\sum m_j a_j}{a_i}} \cdot \prod x_j^{m_j}.$$

Whenever a_i divides $\sum m_j a_j$, this is an element of $k[x_0, \ldots, x_n, x_i^{-1}]_{(0)}$. This way we get an injection

$$k\big[x_0, \ldots, x_n, x_i^{-1}\big]_{(0)} \hookrightarrow k[u_0, \ldots, \hat{u}_i, \ldots, u_n].$$

It is not hard to identify this subring abstractly. Indeed, assume for simplicity that the characteristic of k does not divide a_i and let ζ be a primitive a_ith root of unity. Define an action of the cyclic group $\mathbb{Z}_{a_i}$ on the ring $k[u_0, \ldots, \hat{u}_i, \ldots, u_n]$ by $u_i \mapsto \zeta^{a_j} u_i$. In this case, the ring

$$k\big[x_0, \ldots, x_n, x_i^{-1}\big]_{(0)}$$

is precisely

$$k[u_0, \ldots, \hat{u}_i, \ldots, u_n]^{\mathbb{Z}_{a_i}},$$

the ring of invariants of this action. This allows us to identify the chart U_i as a quotient of affine n-space by the cyclic group $\mathbb{Z}_{a_i}$. The coordinates u_j are called *orbifold coordinates* for the chart U_i.

EXERCISE 3.51. Assume that a_i and a_j are relatively prime. Prove that $U_i \cap U_j$ is smooth. Thus if the a_i are pairwise relatively prime, then the only singular points of $\mathbb{P}(a_0, \ldots, a_n)$ are the "vertices," that is, the points with all but one coordinate zero.

3.52. DIVISORS ON WEIGHTED PROJECTIVE SPACES. The zero set D_F of weighted homogeneous polynomial $F(x_0, \ldots, x_n)$ is a divisor on $\mathbb{P}(a_0, \ldots, a_n)$. Note that D is not necessarily Cartier. Conversely, we claim that every divisor is given this way. This is clear when one of the a_i, say a_0, is one. Indeed, in this case $U_0 \cong \mathbb{A}^n$ and so if $D \subset \mathbb{P}(a_0, \ldots, a_n)$ then $D \cap U_0$ is also a divisor hence given by a polynomial $f(X_1, \ldots, X_n) = 0$. Its weighted homogenization $F(x_0, \ldots, x_n)$ defines D. The only possible problem is with the divisor at infinity, which is however given by the vanishing of x_0. The general case can be done similarly, we do not need it in the sequel.

The weighted degree of F is called the *degree of the divisor D_F*. The degree gives an isomorphism from the divisor class group of Weil divisors modulo linear equivalence to the integers $\mathbb{Z}$. In particular, the set of all weighted homogeneous polynomials of some fixed degree m determines a linear system on $\mathbb{P}(a_0, \ldots, a_n)$, which we denote by $|mH|$. Note that this linear system can be empty for small values of m; for instance H is empty if and only if $a_i \geq 2$ for every i. Nonetheless, the linear system H is called the *hyperplane system* on the weighted projective space.

EXERCISE 3.53. Prove that the canonical class of $\mathbb{P}(a_0, \ldots, a_n)$ is linearly equivalent to $-(\sum a_i)H$, in the case where a_0 and a_1 are both one. (The formula holds in general but the $a_0 = a_1 = 1$ case is easier to prove.)

PROOF OF THEOREM 3.36(5) AND (6). Let X be a degree two Del Pezzo surface. Then $|-K_X|$ is base point free and gives a surjective morphism $X \to \mathbb{P}^2$. Thus $H^0(\mathcal{O}(-K_X)) = \langle x_1, x_2, x_3 \rangle$ generates a six-dimensional subspace of $H^0(\mathcal{O}(-2K_X))$. By Lemma 3.38, $H^0(\mathcal{O}(-2K_X))$ has dimension seven, hence $H^0(\mathcal{O}(-K_X))$ and one more element $x_0 \in H^0(\mathcal{O}(-2K_X))$ generate the anti-canonical ring (given that we know it is generated by elements of degree two and less from Exercise 3.46). The dimension of the space of degree four polynomials in the $x_0, \ldots, x_3$ is 22 whereas $H^0(\mathcal{O}(-4K_X))$ has dimension 21. Thus we obtain one degree four relation.

The arguments for the degree one case are similar so we just sketch them. By computing the dimension of $H^0(\mathcal{O}(-jK_X))$ for $j = 1, 2, 3$, we see that the anti-canonical ring is generated by $H^0(\mathcal{O}(-K_X))$ and one element each of $H^0(\mathcal{O}(-jK_X))$ for $j = 2, 3$. We get a single relation in degree six. □

COROLLARY 3.54. 1. *A degree two Del Pezzo surface (over a field whose characteristic is not two) is birational to an affine surface defined by a polynomial*

$$x^2 + f(y, z)$$

where f is an (inhomogeneous) polynomial of degree at most four.

2. *A degree one Del Pezzo surface (over a field whose characteristic is neither two nor three) is birational to an affine surface defined by a polynomial*

$$x^2 + y^3 + yf(z) + g(z)$$

where f is an (inhomogeneous) polynomial of degree at most four and g is an (inhomogeneous) polynomial of degree at most six.

Conversely, a general choice of f corresponds to a Del Pezzo surface with the expected degree.

PROOF. Consider first the degree two case. Here we have an equation of weighted degree four. If it does not involve x_0^2, then the Del Pezzo surface passes through the singular point $(1 : 0 : 0 : 0) \in \mathbb{P}(2, 1, 1, 1)$, making it singular, a contradiction. Thus dehomogenizing and completing the square (if char $k \neq 2$) we get the required normal form.

In the degree one case, the equation must involve x_0^2 and x_1^3. Otherwise, the Del Pezzo surface passes through one of the singular points $(1:0:0:0) \in \mathbb{P}(3,2,1,1)$ or $(0:1:0:0) \in \mathbb{P}(3,2,1,1)$. As before, this is impossible since X is smooth. If $\operatorname{char} k \neq 2, 3$ then we complete the square and eliminate the x_1^2-terms to get the required normal form. □

EXERCISE 3.55. Consider the projective closure in $\mathbb{P}^3$ of the affine surfaces given in Corollary 3.54. Show how to obtain the original Del Pezzo surfaces by a sequence of blowing ups and downs. (Hint: It may be easier to start with special equations like

$$x^2 + y^4 + z^4 + 1 = 0 \quad \text{and} \quad x^2 + y^3 + z^6 + 1 = 0.$$

You need to resolve the singularities and then contract (-1)-curves.)

This awkward connection between the resulting surfaces in $\mathbb{P}^3$ and the Del Pezzo surface is the main reason why one should work with weighted projective spaces in these examples.

The final exercise of this chapter summarizes the traditional classification of Del Pezzo surfaces over algebraically closed fields.

EXERCISE 3.56. Show that if a Del Pezzo surface is obtained from a smooth surface S by blowing up a point, then S is also Del Pezzo. Using Theorem 3.5, conclude that over an algebraically closed field, every Del Pezzo surface is either $\mathbb{P}^1 \times \mathbb{P}^1$ or a blowup of $\mathbb{P}^2$ in at most eight points.

4

Nonrationality via reduction modulo p

A smooth projective variety is said to be *Fano* if its anti-canonical bundle is ample. In particular, a Fano surface is simply a Del Pezzo surface. In Chapter 3, we saw that every Del Pezzo surface is geometrically rational. Over $\mathbb{C}$, the obvious obstructions to rationality – such as the plurigenera – all vanish for a Fano variety of any dimension. One might wonder whether, as in the surface case, a smooth Fano variety of any dimension is always rational. The purpose of this chapter is to show that, quite to the contrary, there exist an abundance of nonrational Fano varieties of every dimension greater than two.

Our method is based on reduction to prime characteristic, where we make use of the rather special features of differential forms. By its nature, this approach yields statements only about "very general" varieties in certain families, and does not seem to be able to produce statements about, for example, *all* smooth hypersurfaces of a given degree. By contrast, the Noether–Fano method we later develop in Chapters 5 and 6 does yield completely general statements; we use it for instance to prove that no smooth quartic three-fold is rational. On the other hand, the reduction to prime characteristic technique here can be applied in a greater range of situations than the Noether–Fano method. For example, in this chapter we show that certain very general cyclic covers of projective space of any dimension are nonrational Fano varieties; the technique can be adapted to show that very general hypersurfaces of certain degrees are nonrational Fano varieties. This technique is originally due to Kollár (1995); see also Kollár (1996).

In Section 1, we overview our main theorem, as well as the key observation that leads to its proof. This key observation is Proposition 4.6, stating that varieties which admit a big line sub-bundle of differential forms are not ruled. So our goal is to construct varieties that are both Fano and which admit such a big line bundle. The varieties that serve this dual purpose are certain cyclic covers of projective space, so Section 2 is devoted to a careful discussion of cyclic covers, including an analysis of when they are Fano. In Section 3, we construct the needed big line bundles on these cyclic covers, but *only in the case where the ground field has characteristic* p, by exploiting the unusual behavior of differential forms in positive characteristic. Thus in Section 3, we are already able to produce a slew of examples of nonruled (but singular) Fano varieties in positive characteristic.

In order to get characteristic zero examples, we rely on the lifting techniques developed by Matsusaka, which we outline in Section 4. The main proofs are given in Section 5, where we rely on an old theorem of Abhyankar as well as some delicate properties of the relative canonical module in mixed characteristic. The necessary details about relative canonical modules are carried out in Section 6.

Finally, Section 7 contains a series of explicit examples worked out by J. Rosenberg, including explicit examples of smooth Fano varieties defined over the rational numbers that are not geometrically rational.

4.1 Nonrational cyclic covers

The main theorem in this chapter asserts that certain cyclic covers of projective space are nonrational Fano varieties. We discuss cyclic covers in detail in Section 2, but before stating the main result, we briefly recall the definition.

DEFINITION 4.1. The degree *d cyclic cover of* $\mathbb{P}^n$ ramified along the hypersurface D is the normalization of $\mathbb{P}^n$ in the function field

$$k(x_1, \ldots, x_n, \sqrt[d]{f}),$$

where f is an affine equation for D in an affine chart of $\mathbb{P}^n$ with affine coordinates $x_1, \ldots, x_n$.

Recall that the *normalization* of an irreducible variety X in a finitely generated extension L of its function field is, by definition, the variety obtained by patching together the affine schemes Spec B_i where B_i is the integral closure of A_i in L, where the schemes Spec A_i form an open affine cover of X.

We can now state the main theorem of this chapter. The entire chapter is devoted to a careful proof of this theorem.

THEOREM 4.2 (Kol95). *There exists a nonrational smooth complex Fano variety of every dimension n greater than two. Specifically, if p is a prime number, then any degree p cyclic cover of projective n-space ramified over a very general hypersurface of degree mp is a nonrational smooth projective*

Fano variety, provided that mp is in the range

$$n+1 < mp < n+1+\frac{n+1}{p-1}.$$

A related result that can be proved by the same methods is the following.

THEOREM 4.3. *A very general complex hypersurface of degree d in $\mathbb{P}^{n+1}$ is a nonrational Fano variety, provided that d is in the range*

$$\frac{2}{3}n+3 \leq d \leq n+1.$$

By a *very general hypersurface* in both theorems above we mean a hypersurface whose defining equation is outside a countable union of Zariski closed subsets of the vector space of all polynomials of that degree in the corresponding polynomial ring. In fact, it is quite likely that every smooth degree p cyclic cover of projective space ramified over a hypersurface of degree mp, as well as every smooth degree d hypersurface in $\mathbb{P}^{n+1}$ (with mp and d in the stated ranges) is nonrational, but this does not seem to be provable by the methods here.

One can use the proof of Theorem 4.2 to write down explicit examples of smooth Fano varieties over the rational numbers that are not rational over $\mathbb{C}$; see Section 7. Similarly, one can find explicit equations for nonrational Fano hypersurfaces over $\mathbb{Q}$. Note, however, that the existence of examples over $\mathbb{Q}$ does not follow immediately from Theorem 4.2 as stated: there may be no sufficiently general cyclic covers defined over a countable field.

This chapter is devoted to proving Theorem 4.2. By paying careful attention to the details, one actually comes to the stronger conclusion that the constructed cyclic covers are not ruled. To keep technicalities to a minimum here, we content ourselves with the weaker conclusion, and refer the interested reader to Kollár (1996, V.5).

We prove Theorem 4.2 by reduction to positive characteristic. Indeed, we first prove the following slightly more precise theorem in characteristic $p > 0$:

THEOREM 4.4. *Fix positive integers m and n and a prime number p satisfying $(p-1)m < n+1 < pm$. Let F be a homogeneous polynomial in $n+1$ variables over a field k of characteristic p, and assume that F satisfies a non-degeneracy condition to be given later in Assumption 4.19. Then the degree p cyclic cover of $\mathbb{P}^n$ ramified over the hypersurface defined by F is a (singular) Fano variety that is not ruled, for all $n \geq 3$.*

The Fano varieties arising in Theorem 4.4 are never smooth. Although the classical definition of a Fano variety requires smoothness, recent usage frequently allows singular varieties. (We of course have to assume that the anti-canonical divisor is Cartier, or at least $\mathbb{Q}$-Cartier for ampleness to make sense.)

We use Theorem 4.4, which is essentially a prime characteristic version of Theorem 4.2, to prove Theorem 4.2 by finding a flat family of Fano cyclic covers whose special fiber is of prime characteristic p and whose generic fiber is of characteristic zero. Although the special fiber is singular, the generic fiber will be a smooth Fano variety. By Theorem 4.4, we will know that the special fiber is not ruled. We then use a result of Matsusaka, Theorem 4.23, stating that ruledness is well-behaved in families to conclude that the generic fiber is not ruled. Note that there is no analogous theorem about the behavior of rationality in families; we are really forced to consider ruledness in Theorem 4.4. See Caution 4.24.

The key point at the heart of the method is that the existence of certain big line bundles of differential forms is an obstruction to ruledness. In prime characteristic, we directly construct such big line bundles on our cyclic covers, leading to the proof of Theorem 4.4 in Section 3. These big line bundles do not exist in the characteristic zero case, but nonetheless, Matsusaka's result enables us to "lift" our examples to characteristic zero. We now discuss this key point concerning big line bundles and ruledness.

Recall that a big line bundle is, intuitively speaking, one that is "birationally ample;" the precise definition follows.

DEFINITION 4.5. A line bundle on a variety is *big* if the complete linear system associated to some positive power defines a birational map onto its image.

Recall also that X is separably uniruled if there is a separable, generically finite map $Y \times \mathbb{P}^1 \dashrightarrow X$, where Y is some arbitrary variety; see Definition 1.57. We can now state the key proposition.

PROPOSITION 4.6. *A smooth projective variety is not separably uniruled if it admits a big line bundle contained in some vector bundle of differential forms* $\bigwedge^i \Omega_X$, *for some* i.

REMARK 4.7. If X is a smooth projective variety of characteristic zero, then the Bogomolov–Sommese vanishing theorem guarantees that the only sheaf of differential forms that can contain *any* big line bundle is the line bundle $\bigwedge^{\dim X} \Omega_X$ itself; see Esnault and Viehweg (1992, p. 58). In particular, in characteristic zero, Proposition 4.6 degenerates to the following simple statement: no smooth projective variety of general type is uniruled. Indeed, by definition, a variety is of general type if its canonical bundle is big. (The fact that such a variety can not be uniruled is easy in any case, because obviously the higher plurigenera can not all vanish; see Theorem 1.58.)

Proof of Proposition 4.6. Suppose that X is separably uniruled, and let $\phi : Y \times \mathbb{P}^1 \dashrightarrow X$ be a separable, generically finite map. Without loss of generality, Y may be replaced by a smooth affine open subset on which ϕ is a morphism. The pull-back map on differential forms

$$\phi^* : \phi^*\Omega_X \to \Omega_{Y\times\mathbb{P}^1}$$

is an isomorphism on a dense open set over which ϕ is smooth. In fact, all we need is that ϕ^* is injective, so that a subsheaf $\mathcal{M}$ of $\bigwedge^i \Omega_X$ pulls back to a subsheaf $\phi^*\mathcal{M}$ of $\bigwedge^i \Omega_{Y\times\mathbb{P}^1}$.

On the other hand,

$$\Omega_{Y\times\mathbb{P}^1} \cong \pi_1^*\Omega_Y \oplus \pi_2^*\Omega_{\mathbb{P}^1} \cong \pi_1^*\Omega_Y \oplus \pi_2^*\mathcal{O}_{\mathbb{P}^1}(-2),$$

where π_i are the coordinate projections. Thus we have

$$\phi^*\mathcal{M}^{\otimes m} \subset \big(\textstyle\bigwedge^i(\Omega_{Y\times\mathbb{P}^1})\big)^{\otimes m} \cong \big(\textstyle\bigwedge^i(\pi_1^*\Omega_Y \oplus \pi_2^*\mathcal{O}_{\mathbb{P}^1}(-2))\big)^{\otimes m}.$$

If $\mathcal{M}$ is big, so is its pull-back under any generically finite map so, in particular, the powers of $\phi^*\mathcal{M}$ have enough global sections to separate points on an open set of $Y \times \mathbb{P}^1$. But this is impossible, since

$$H^0\big(Y\times\mathbb{P}^1, \big(\textstyle\bigwedge^i(\pi_1^*\Omega_Y \oplus \pi_2^*\mathcal{O}_{\mathbb{P}^1}(-2))\big)^{\otimes m}\big) = H^0\big(Y\times\mathbb{P}^1, \big(\textstyle\bigwedge^i\pi_1^*\Omega_Y\big)^{\otimes m}\big),$$

and these sections do not separate the points in the fibers $\{y\} \times \mathbb{P}^1$ from each other. ☐

Thus our goal in Section 3 is to show that for the cyclic covers of the type described in Theorem 4.4, we can find a big line bundle of differential forms. For this, we need a careful understanding of differentiation in prime characteristic. First, in the next section, we recall some basic properties of cyclic covers.

4.2 Construction of cyclic covers

Consider a degree p cyclic cover Z of projective space, ramified along some divisor D, as defined in Definition 4.1. If the divisor D is reduced, then the cyclic cover is given by affine equations $y^p = f(x_1, \ldots, x_n)$ over each affine chart of $\mathbb{P}^n$, where $x_1, \ldots x_n$ are local coordinates on the corresponding chart and $f(x_1, \ldots, x_n)$ is a local defining equation for D there. In particular, there is a natural map

$$Z \to \mathbb{P}^n$$

which is indeed a degree p cover of projective space. If the characteristic of the ground field does not divide p, one easily verifies that the preimage of each point consists of p distinct points, except over D, where the preimage is a single point of multiplicity p. Thus with this restriction on the characteristic, $Z \to \mathbb{P}^n$ is indeed a degree p cover ramified along D. On the other hand, if the characteristic divides p, then the map is ramified everywhere. Nonetheless, we abuse terminology by referring to Z as a "degree p cyclic cover of $\mathbb{P}^n$ ramified along D" even in this case.

Thus cyclic covers of projective space are projective models of an affine hypersurface defined by an equation $y^p = f(x_1, \dots x_n)$. In this section, we discuss some concrete ways to realize these projective models, especially when the degree of the divisor D is a multiple of p. The obvious projective model, namely the closure in $\mathbb{P}^{n+1}$, is not useful because it is never smooth, as the next exercise shows.

EXERCISE 4.8. Prove that the projective closure in $\mathbb{P}^{n+1}$ of the affine hypersurface defined by $y^d - f$ is never smooth if the degree of f is greater than $d + 1$.

Next, we consider smoothness of cyclic covers.

EXERCISE 4.9. Prove that the nonsmooth points of the affine hypersurface defined by $y^p - f$ are described in terms of the critical points of f as follows. In the case where the characteristic of the ground field divides p, the nonsmooth points are in one-to-one correspondence with the critical points of f, with $(y, x_1, \dots, x_n)$ a nonsmooth point if and only if $(x_1, \dots, x_n)$ is a critical point of f. Otherwise, the point $(y, x_1, \dots, x_n)$ is not smooth if and only if $(x_1, \dots, x_n)$ is a critical point of f and $f(x_1, \dots, x_n) = 0$.

Show that for sufficiently general choice of f, the hypersurface $y^p - f$ has only isolated nonsmooth points in the characteristic p case, and is everywhere smooth in characteristic zero case. In particular, show that a cyclic cover of projective space ramified along a smooth divisor D is smooth, provided the characteristic of the ground field does not divide the degree of the cover.

We now describe two explicit projective models for our cyclic covers that are useful for later computations.

4.10. CYCLIC COVERS AS HYPERSURFACES IN WEIGHTED PROJECTIVE SPACE. Let $\mathbf{P} := \mathbb{P}(m, 1, 1, \dots, 1)$ be the weighted projective space with coordinates $Y, X_0, X_1, \dots, X_n$, where Y has degree m and the X_i all have degree one, as defined in Section 3.48. Note that $\mathbf{P}$ has an isolated singular point at $(1 : 0 : \dots : 0)$.

Consider a hypersurface D in $\mathbb{P}^n$ defined by a polynomial F of degree mp in the homogeneous coordinates $X_0, \ldots, X_n$. Let Z be the closed subscheme of the weighted projective space defined by the weighted homogeneous polynomial $Y^p - F$. It is not hard to check that the natural map

$$Z \to \mathbb{P}^n$$

makes Z into a degree p cyclic cover of projective space ramified along D. In particular, for smooth D, the cyclic cover Z is smooth (at least when the characteristic does not divide p).

4.11. Cyclic covers as subvarieties of a line bundle. Cyclic covers can also be described as closed subschemes of the total space of a certain line bundle on projective space.

Let $X_0, X_1, \ldots, X_n$ be homogeneous coordinates for $\mathbb{P}^n$. For each i, let V_i be the open affine subset where X_i does not vanish. The affine coordinates for V_i are $\frac{X_j}{X_i}$ for $j = 0, 1, \ldots n$, excepting $j = i$.

Consider the line bundle $\mathcal{O}_{\mathbb{P}^n}(m)$ on $\mathbb{P}^n$. This line bundle is trivialized on the cover $\{V_i\}_{i=0}^n$ of $\mathbb{P}^n$. Fixing generators s_i on V_i, we have patching data for $\mathcal{O}_{\mathbb{P}^n}(m)$ on $V_i \cap V_j$

$$s_i = \left(\frac{X_i}{X_j}\right)^m s_j.$$

Let U be the variety formed by the union of the open sets $U_i = V_i \times \mathbb{A}^1$, patched together by the relations $y_i = (\frac{X_i}{X_j})^{-m} y_j$, where y_i is the local coordinate for the copy of $\mathbb{A}^1$ in U_i. The natural projection $\pi : U \to \mathbb{P}^n$ defines an $\mathbb{A}^1$-bundle over $\mathbb{P}^n$. This is the total space of the line bundle whose sheaf of sections is $\mathcal{O}_{\mathbb{P}^n}(m)$; it is neither affine nor projective.

Now let $F(X_0, \ldots, X_n)$ be the homogeneous polynomial of degree mp defining the divisor D of $\mathbb{P}^n$, and let Z be the subvariety of U defined locally by the equations $y_i^p - \frac{F}{X_i^{mp}}$ in the open subset U_i. In each U_i, the variety Z has exactly the form of the affine hypersurface $y^p = f$. Furthermore, there is a natural projection

$$Z \to \mathbb{P}^n,$$

obtained by restricting the natural structural map of the line bundle $U \to \mathbb{P}^n$; abusing notation, we denote both these maps by π. Again, $Z \to \mathbb{P}^n$ is a finite surjective map, of degree p, ramified along the divisor D (in case p is not divisible by the characteristic) or everywhere (otherwise). In fact, it is easy to see that this scheme Z is isomorphic to the subscheme Z of the weighted projective space defined in §4.10, and that the corresponding projections to $\mathbb{P}^n$

coincide as well. Thus this scheme is also a concrete realization of the degree p cyclic cover of projective space ramified along D.

The following observation will be useful later.

LEMMA 4.12. *There is a natural isomorphism $\mathcal{O}_U(-Z) = \pi^*\mathcal{O}_{\mathbb{P}^n}(-mp)$.*

PROOF. Indeed, the patching data for $\mathcal{O}_U(-Z)$, the defining ideal for Z as a closed subvariety of U, have the same transition functions as $\pi^*\mathcal{O}_{\mathbb{P}^n}(-mp)$: a local generator for either sheaf on the affine neighborhood U_j is transformed into a local generator on U_i by multiplication by $(\frac{X_i}{X_j})^{-mp}$. $\square$

4.13. FANO CYCLIC COVERS. For the appropriate choices of the integers n, m and p, the cyclic covers we have constructed are Fano.

PROPOSITION 4.14. *The anti-canonical sheaf of a degree p cyclic cover of projective n-space ramified along a divisor D of degree mp is ample whenever*

$$mp - m < n + 1.$$

This holds whether or not Z is smooth.

PROOF. There are several different ways to compute the canonical sheaf $\omega_Z \cong \mathcal{O}_Z(K_Z)$. We explain the least succinct way first, because the computation is of the most use later.

Method 1: We view Z as a divisor in the total space U as in §4.11, and use the adjunction formula to compute the canonical sheaf of Z.

First compute ω_U using the exact sequence

$$0 \to \pi^*\Omega_{\mathbb{P}^n} \to \Omega_U \to \pi^*\mathcal{O}_{\mathbb{P}^n}(-m) \to 0. \tag{4.14.1}$$

The exactness is easily verified by the local computation

$$ds_i = d\left[\left(\frac{X_i}{X_j}\right)^{-m} s_j\right] = \left(\frac{X_i}{X_j}\right)^{-m} ds_j + d\left[\left(\frac{X_i}{X_j}\right)^{-m}\right] s_j,$$

observing that $d[(\frac{X_i}{X_j})^{-m}]$ is pulled back from $\Omega_{\mathbb{P}^n}$ and that $(\frac{X_i}{X_j})^{-m} ds_j$ maps to a local generator $\pi^*\mathcal{O}_{\mathbb{P}^n}(-m)$. Therefore,

$$\omega_U = \bigwedge^{n+1}\Omega_U = \left(\bigwedge^n \pi^*\Omega_{\mathbb{P}^n}\right) \otimes \pi^*\mathcal{O}_{\mathbb{P}^n}(-m) \cong \pi^*\mathcal{O}_{\mathbb{P}^n}(-n-1-m).$$

By adjunction, therefore,

$$\begin{aligned} \omega_Z \cong (\omega_U \otimes \mathcal{O}_U(Z))|_Z &\cong \pi^*\mathcal{O}_{\mathbb{P}^n}(-n-1-m) \otimes \pi^*\mathcal{O}_{\mathbb{P}^n}(mp) \\ &\cong \pi^*\mathcal{O}_{\mathbb{P}^n}(mp-n-1-m), \end{aligned}$$

where π also denotes the restriction of π to Z. Because $\pi : Z \to \mathbb{P}^n$ is a finite map, the pull back of an ample line bundle on $\mathbb{P}^n$ is ample. This proves that ω_Z^{-1} is ample whenever the numerical condition $mp - m < n + 1$ is satisfied.

Method 2: We view Z as a hypersurface in the weighted projective space $\mathbf{P}$ as in §4.10 and use the adjunction formula. This yields $K_Z = (K_{\mathbf{P}} + Z)|_Z$, so that $K_Z \sim (-m - n - 1 + mp)H|_Z$; see Exercise 3.53. All this works well since Z does not pass through the singular point of $\mathbf{P}$. It follows that $-K_Z$ is ample if and only if $mp - m < n + 1$.

Method 3: We view Z as a finite cover of $\mathbb{P}^n$ and use the Hurwitz formula. This formula says that the canonical class for Z is the pullback of the canonical class of $\mathbb{P}^n$ plus the appropriate multiple of the ramification divisor. That is,

$$K_Z \equiv \pi^*\big(K_{\mathbb{P}^n} + \tfrac{p-1}{p} D\big) \equiv ((p-1)m - n - 1)\pi^* H,$$

where H is the hyperplane class of $\mathbb{P}^n$. This method, however, can not be applied when the characteristic divides p, because then $Z \to \mathbb{P}^n$ is everywhere ramified. □

4.3 Differential forms in characteristic *p*

Our goal is to apply Proposition 4.6 to conclude that the special cyclic covers constructed in the previous section are not ruled. In order to do so, we try to find a big line sub-bundle of a sheaf of differential forms on the cyclic cover Z. Although we do not quite succeed, we eventually find such a big line bundle on a desingularization of Z. This will be constructed from an exterior power of a special subsheaf Q of Ω_Z, which we now construct.

4.15. A SPECIAL SUBSHEAF OF KÄHLER DIFFERENTIALS IN CHARACTERISTIC *p*. Let p be a prime number, and let Z be a degree p cover of $\mathbb{P}^n$ ramified along a divisor D of degree mp.[1] We consider Z as a closed subscheme the total space U of the line bundle $\mathcal{O}_{\mathbb{P}^n}(m)$ as explained in §4.11.

Consider the familiar exact sequence

$$\mathcal{O}_U(-Z)|_Z \xrightarrow{d} \Omega_U|_Z \to \Omega_Z \to 0. \tag{4.15.1}$$

(This is the "conormal" or the "second exact sequence;" see Matsumura (Mat80, Thm. 58, p. 187). In the hypersurface case that we use, this sequence is

[1] We remind the reader that, despite our choice of terminology, the variety Z is in fact everywhere ramified over $\mathbb{P}^n$ in characteristic p.

equivalent to the computations done in Shafarevich (1994, III.6.4).) Let us scrutinize the map d. In the chart $U_0 = \{X_0 \neq 0\}$, set $x_i = \frac{X_i}{X_0}$, $y = \frac{Y}{X_0}$, and $\frac{F}{X_0^{mp}} = f(x_1, \ldots, x_n)$. The map

$$d : \mathcal{O}_U(-Z)|_Z \to \Omega_U|_Z$$

sends the local generator $y^p - f(x_1, \ldots, x_n)$ to

$$d(y^p - f(x_1, \ldots, x_n)) = -\frac{\partial f}{\partial x_1} dx_1 - \cdots - \frac{\partial f}{\partial x_n} dx_n + p y^{p-1} dy.$$

Something very interesting happens in characteristic p: the image of d is contained in the subsheaf of $\Omega_U|_Z$ generated by the differentials $dx_1, \ldots, dx_n$, that is,

$$d(\mathcal{O}_U(-Z)|_Z) \subset \pi^* \Omega_{\mathbb{P}^n}.$$

Making use of the $\mathcal{O}_Z$-module isomorphism $\mathcal{O}_U(-Z)|_Z \cong \pi^* \mathcal{O}_{\mathbb{P}^n}(-mp)$ proved in Lemma 4.12, we can define an $\mathcal{O}_Z$-module map *in characteristic p only*

$$d : \pi^* \mathcal{O}_{\mathbb{P}^n}(-mp) \to \pi^* \Omega_{\mathbb{P}^n} \tag{4.15.2}$$

sending a local generator f to $df = \sum \frac{\partial f}{\partial x_i} dx_i$ and extending $\mathcal{O}_Z$-linearly. The use of the symbol d to denote this map is somewhat misleading, since the map is not a derivation, but is $\mathcal{O}_Z$-linear. Miraculously, this is a well defined $\mathcal{O}_Z$-module map in characteristic p, because the transition functions for $\mathcal{O}_{\mathbb{P}^n}(-mp)$ are pth powers, and are therefore killed by d.

Let Q be the cokernel of the $\mathcal{O}_Z$-module map $d : \pi^* \mathcal{O}_{\mathbb{P}^n}(-mp) \to \pi^* \Omega_{\mathbb{P}^n}$. There is an exact sequence of $\mathcal{O}_Z$-modules

$$\pi^* \Omega_{\mathbb{P}^n} \to \Omega_U|_Z \to \pi^* \mathcal{O}_{\mathbb{P}^n}(-m) \to 0. \tag{4.15.3}$$

obtained by restricting the sequence (4.14.1) to Z. We can combine this with the exact sequence (4.15.1) in the commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \\
 & & \pi^*\mathcal{O}_{\mathbb{P}^n}(-m) & & \pi^*\mathcal{O}_{\mathbb{P}^n}(-m) & & \\
 & & \uparrow & & \uparrow & & \\
\pi^*\mathcal{O}_{\mathbb{P}^n}(-mp) & \to & \Omega_{U|Z} & \to & \Omega_Z & \to & 0 \\
 & & \uparrow & & \uparrow & & \\
\pi^*\mathcal{O}_{\mathbb{P}^n}(-mp) & \to & \pi^*\Omega_{\mathbb{P}^n} & \to & Q & \to & 0 \\
 & & & & \uparrow & & \\
 & & & & 0 & &
\end{array}$$

From the diagram, we get an exact sequence of $\mathcal{O}_Z$-modules:

$$0 \to Q \to \Omega_Z \to \pi^*\mathcal{O}_{\mathbb{P}^n}(-m) \to 0. \tag{4.15.4}$$

An exterior power of Q will give us the desired big sub-bundle of a sheaf of differential forms (at least after desingularizing). It is important to realize that the assumption that k has characteristic p is essential: nothing like this is possible in characteristic zero.

The next exercise is not essential for our computation, but it should help clarify what is going on in the construction of Q.

EXERCISE 4.16. Let X be an arbitrary variety over a field k. A *connection* on an invertible sheaf $\mathcal{L}$ of $\mathcal{O}_X$-modules is a k-linear map

$$\nabla : \mathcal{L} \to \mathcal{L} \otimes \Omega_X$$

satisfying $\nabla(fs) = f\nabla(s) + s \otimes df$ for local sections $s \in \mathcal{L}$ and $f \in \mathcal{O}_X$.

1. Explain how a connection can be interpreted as a rule for differentiating sections of line bundles.
2. Show that if k has prime characteristic p, then any line bundle that is a pth power admits a connection.
3. Observe that the map d above in (4.15.2) can be constructed from the composition

$$\mathcal{O} \xrightarrow{\text{mult by } s} \mathcal{L}^p \xrightarrow{\nabla} \mathcal{L}^p \otimes \Omega_X$$

where s is a global section of $\mathcal{L}^p$, using the identifications

$$H^0(\mathcal{L}^p \otimes \Omega_X) = \mathrm{Hom}(\mathcal{O}_X, \mathcal{L}^p \otimes \Omega_X) = \mathrm{Hom}(\mathcal{L}^{-p}, \Omega_X).$$

4.17. BIGNESS OF THE SPECIAL SUB-BUNDLE OF DIFFERENTIAL FORMS. With an eye towards applying Proposition 4.6, we hope to find integers m, n and p so that $\bigwedge^{n-1} Q$ is a big invertible sheaf. Assume, for a moment, that Z is smooth. The sequence (4.15.4) would imply that Q is locally free of rank $n-1$, and that $\bigwedge^{n-1} Q \hookrightarrow \bigwedge^{n-1} \Omega_Z$. We could easily determine the range of values for m, n and p for which $\bigwedge^{n-1} Q$ is big. Indeed,

$$\omega_Z = \bigwedge^n \Omega_Z = \bigwedge^{n-1} Q \otimes \pi^*\mathcal{O}_{\mathbb{P}^n}(-m),$$

so that

$$\bigwedge^{n-1} Q = \omega_Z \otimes \pi^*\mathcal{O}_{\mathbb{P}^n}(m) = \pi^*\mathcal{O}_{\mathbb{P}^n}(mp - n - 1),$$

using the isomorphism $\omega_Z = \pi^*\mathcal{O}_{\mathbb{P}^n}(mp - m - n - 1)$ verified in Proposition 4.14. From this we could conclude that $\bigwedge^{n-1} Q$ is ample (and hence big) whenever $n + 1 < mp$.

Because there are plenty of choices of integers m, p and n for which the constraints

$$mp - m < n + 1 < mp$$

hold, we could then deduce by Proposition 4.6 that the corresponding cyclic covers are nonrational Fano varieties. Unfortunately, however, this argument fails because the variety Z is *virtually never* smooth: this is the price to be paid for the characteristic p trickery that allowed us to construct Q. Indeed, $\bigwedge^{n-1} Q$ is not even an invertible sheaf in general.

It is easy to alter $\bigwedge^{n-1} Q$ so as to get a big invertible sheaf. On the smooth locus of Z, the sheaf $\bigwedge^{n-1} Q$ is naturally isomorphic to $\pi^*\mathcal{O}_{\mathbb{P}^n}(mp - n - 1)$. Assume that Z is normal, so that any invertible sheaf defined on the complement of a codimension two closed subscheme extends *uniquely* to a reflexive sheaf of $\mathcal{O}_Z$-modules. Since $\bigwedge^{n-1} Q$ agrees with $\pi^*\mathcal{O}_{\mathbb{P}^n}(mp - n - 1)$ on the smooth locus, its "reflexive hull"

$$(\textstyle\bigwedge^{n-1} Q)^{**} = \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{H}om_{\mathcal{O}_Z}(\textstyle\bigwedge^{n-1} Q, \mathcal{O}_Z), \mathcal{O}_Z)$$

is an invertible sheaf of $\mathcal{O}_Z$-modules isomorphic to $\pi^*\mathcal{O}_{\mathbb{P}^n}(mp - n - 1)$; that is

$$\left(\bigwedge^{n-1} Q\right)^{**} \cong \pi^*\mathcal{O}_{\mathbb{P}^n}(mp - n - 1).$$

This sheaf is ample when $mp > n + 1$, and is a subsheaf of $(\bigwedge^{n-1} \Omega_Z)^{**}$. On the smooth locus of Z, it restricts to an invertible sheaf of differential $n - 1$ forms on Z. Thus our strategy is to resolve the singularities of Z, and try to apply Proposition 4.6 to the pull-back of $(\bigwedge^{n-1} Q)^{**}$.

4.18. Desingularizing Z. Let Z be the degree p cyclic cover of projective space ramified along the divisor D of degree mp. In order to apply Proposition 4.6 to conclude that Z is not ruled, we must resolve the singularities of Z. Bigness is preserved under birational pull-back, so the pull-back of $(\bigwedge^{n-1} Q)^{**}$ to any desingularization is still a big invertible sheaf. But we must check that this pull-back is a subsheaf of some sheaf of differential forms. We accomplish this by choosing the divisor D so as to make an explicit resolution straightforward.

Let F be the homogeneous defining equation for the divisor D on $\mathbb{P}^n$. Recall that the nonsmooth points of Z are given precisely by the critical points of the dehomogenized polynomials f_i obtained by the substitution $X_i = 1$; see Exercise 4.9. It is easy to desingularize Z under the the following nondegeneracy assumption of F:

ASSUMPTION 4.19. Each dehomogenization

$$F(X_0, \dots, X_{i-1}, 1, X_{i+1}, \dots, X_n) \qquad \text{for } i = 0, \dots, n$$

of F is a polynomial with only nondegenerate critical points.

As usual, a *critical point* of a polynomial f is a point where all the partial derivatives of f vanish, and the critical point is *nondegenerate* if the determinant of the Hessian matrix of second derivatives does not vanish there. Here, "point" means point defined over the algebraic closure of the ground field. Such F exist over any infinite field (with some exceptions in characteristic two); see Exercise 4.22.

Assuming that F has the nondegeneracy condition described above, we complete the proof of Theorem 4.4 by desingularizing Z and verifying that $(\bigwedge^{n-1} Q)^{**}$ pulls back to a subsheaf of regular differential forms.

The advantage of nondegenerate critical points is that, after possibly enlarging the ground field, the affine equation of the hypersurface Z can be assumed of the the form

$$y^p = c + x_1x_2 + x_3x_4 + \cdots + x_{n-1}x_n + \ f_3 \quad \text{if } n \text{ is even, or}$$
$$y^p = c + x_1x_2 + x_3x_4 + \cdots + x_{n-2}x_{n-1} + x_n^2 + f_3 \quad \text{if } n \text{ is odd and } p \neq 2.$$

where the y, x_is are local coordinates at a nonsmooth point of Z, c is a constant, and $f_3 = f_3(x_1, \dots, x_n)$ is a polynomial of order three or more in the x_i. Fortunately, desingularizing such a hypersurface is easy.

EXERCISE 4.20. Show that if f has only isolated nondegenerate critical points, then the affine hypersurface defined by $y^p - f$ becomes smooth upon repeatedly blowing up each nonsmooth point (over the algebraic closure of the ground field), regardless of the characteristic of the ground field.

4.21. PULLING BACK TO A DESINGULARIZATION. Let Z be a degree p cyclic cover of projective space ramified over a divisor D of degree mp which satisfies the nondegeneracy Assumption 4.19.

Having shown that Z can be smoothed by blowing up points, let $q : Z' \to Z$ be this desingularization of Z. Assuming that the ground field is characteristic p, consider the sheaf

$$\mathcal{M} = q^*(\textstyle\bigwedge^{n-1} Q)^{**} = q^*\pi^*\mathcal{O}_{\mathbb{P}^n}(mp - n - 1),$$

where Q is the cokernel of the map (4.15.2) as discussed in §4.17. We have shown that $\mathcal{M}$ is big whenever $mp > n + 1$, and we wish to show that it is contained in $\bigwedge^{n-1} \Omega_{Z'}$. This is just a matter of computing local generators for $\mathcal{M}$ and comparing them to local generators for $\bigwedge^{n-1} \Omega_{Z'}$.

We return to the somewhat mysterious definition of Q. Recall that Q is the cokernel of the very special $\mathcal{O}_Z$-module map

$$d : \pi^*\mathcal{O}_{\mathbb{P}^n}(-mp) \to \pi^*\Omega_{\mathbb{P}^n},$$

defined in (4.15.2). Think of d as the pull-back of a map of $\mathcal{O}_{\mathbb{P}^n}$-modules $d' : \mathcal{O}_{\mathbb{P}^n}(-mp) \to \Omega_{\mathbb{P}^n}$, sending the local generator f to

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i.$$

(We reiterate that this map is deceptively subtle: its existence is a very special consequence of the fact that the ground field has characteristic $p > 0$.)

Taking the $(n-1)$st exterior power of the cokernel Q of $d = \pi^* d'$, we have convenient local generators

$$\eta_i = (-1)^i \frac{dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n}{\partial f / \partial x_i}$$

for $\bigwedge^{n-1} Q$ on the open set where $\partial f / \partial x_i$ is nonzero. Note that $\eta_i = \eta_j$ whenever both are defined. The locus where no η_i is defined is precisely the nonsmooth locus of f. Since this set has codimension at least two, this sheaf extends uniquely to a sheaf $(\bigwedge^{n-1} Q)^{**}$ on all of Z. The extension can be defined as a subsheaf of the constant sheaf of rational differential forms on Z generated by the η_i. By definition of $\mathcal{M}$, these pull back to local generators of $\mathcal{M}$ on the desingularization Z'.

To check that $\mathcal{M} \subset \bigwedge^{n-1} \Omega_{Z'}$, we only need check what happens along the exceptional fibers of $Z' \to Z$, since we already know that the inclusion holds on the smooth locus of Z. This is a straightforward computation; we work it out in one case below.

Let $y, x_1, \ldots, x_n$ be local coordinates for Z near a nonsmooth point, and let $y, x'_1, \ldots, x'_n$ denote local coordinates on the blowup Z' of the ideal $(y, x_1, \ldots, x_n)$, with $x_i = y x'_i$. Computing the pull-back of, say, η_n when n is even and p is 2, we have

$$q^* \eta_n = \frac{d(yx'_1) \wedge \cdots \wedge d(yx'_{n-1})}{\partial (y^2 + x_1 x_2 + \cdots + x_{n-1} x_n + g) / \partial x_n}$$

where g has order 3 or more in $(y, x_1, \ldots, x_n)$. Performing the differentiation, we see that the denominator is $x_{n-1} + h$, where h is order 2 or more in $(y, x_1, \ldots, x_n)$, which we write as $y(x'_{n-1} + y h')$ in local coordinates on the blowup, with h' in $(y, x'_1, \ldots, x'_n)$. Thus

$$\begin{aligned} q^*\eta_n &= \Bigg[y^{n-1}(dx'_1 \wedge \cdots \wedge dx'_{n-1}) \\ &\quad + \sum_{j=1}^{n-1} y^{n-2}(dx'_1 \wedge \cdots \wedge dy \wedge \cdots \wedge dx'_{n-1}) \Bigg] \Bigg/ [y(x'_{n-1} + yh')] \\ &= y^{n-3} \Bigg\{ \Bigg[y(dx'_1 \wedge \cdots \wedge dx'_{n-1}) \\ &\quad + \sum_{j=1}^{n-1} (dx'_1 \wedge \cdots \wedge dy \wedge \cdots \wedge dx'_{n-1}) \Bigg] \Bigg/ [(x'_{n-1} + yh')] \Bigg\}, \end{aligned}$$

where the jth term in the sum has dy in the jth position.

To check that the local generator $q^*\eta_n$ has no pole along the exceptional fibers, we can compute in any open set which intersects the exceptional divisor E. In the neighborhood considered above, the exceptional divisor E is defined by $y = 0$, so the generator $q^*\eta_n$ vanishes along E to at least order $n - 3$. So $\mathcal{M}$ has no poles along E whenever $n \geq 3$. Because the computation along each exceptional divisor is essentially the same, we conclude that $\mathcal{M}$ is a subsheaf of $\bigwedge^{n-1} \Omega_{Z'}$, whenever $n \geq 3$.

Finally, we are in a position to pull all this together to apply Proposition 4.6 to give a proof of our main result in prime characteristic.

Proof of Theorem 4.4. We have just proved in §4.21 that that a desingularization Z' of Z carries the invertible sheaf $\mathcal{M}$ that is a subsheaf of a sheaf of differential forms on Z'. Furthermore, we also saw that $\mathcal{M}$ is big whenever $mp > n + 1$. By Proposition 4.6, we therefore conclude that Z' cannot be separably uniruled when $mp > n + 1$. But because Z' is birationally equivalent to Z, it follows also that Z is not separably uniruled. In particular, Z is not ruled and certainly not rational. On the other hand, for $mp - m < n + 1$, we also proved (see Proposition 4.14) that the cyclic cover Z is a Fano variety. Thus the proof of Theorem 4.4 is complete. □

We have not yet established that Theorem 4.4 ever applies. Namely, we need to check that Assumption 4.19 can be satisfied. First we prove that this assumption holds for a non-empty Zariski open subset of all polynomials of a given degree. This proves existence over infinite fields.

Exercise 4.22. A critical point P of a polynomial f is non-degenerate if the determinant of the Hessian matrix $(\frac{\partial^2 f}{\partial x_i \partial x_j})$ does not vanish at P, or equivalently, if $\{\frac{\partial f}{\partial x_i}\}_{i=1}^n$ generate the maximal ideal of P. Prove the following Morse lemma for polynomials over an infinite field k.

1. If the characteristic of k is greater than two, then a general polynomial function of degree d in n variables over k has only non-degenerate critical points.
2. If k has characteristic two, then every critical point of a polynomial in an odd number of variables is degenerate, whereas the general polynomial function of an even number of variables has only non-degenerate critical points.

When the ground field is infinite, the Morse lemma ensures that a sufficiently general polynomial F satisfies Assumption 4.19. It is not obvious that such F exist over finite fields. On the other hand, Proposition 4.31 gives explicit examples of polynomials F over any finite field satisfying Assumption 4.19 and also the numerical constraints $mp - m < n + 1 < pm$. A specific example, over any field of characteristic p, is the polynomial

$$\sum_{i=0}^{n} X_i^{mp-1} X_{i+1},$$

where the subscripts are taken modulo $n+1$. Here $n \not\equiv -1 \mod p$ is any integer greater than two satisfying $pm - m < n + 1 < pm$.

4.4 Reduction to characteristic p

We now explain how to deduce Theorem 4.2 from Theorem 4.4 by reducing modulo p.

To get a rough idea how this is done, suppose first that we wish to construct an example defined over $\mathbb{Q}$. Let F be a homogeneous polynomial in $\mathbb{Z}[X_0, \ldots, X_n]$ of degree mp. Assign degrees $\deg Y = m$ and $\deg X_i = 1$ and consider the scheme

$$Z_{\mathbb{Z}} = \operatorname{Proj} \mathbb{Z}[Y, X_0, \ldots, X_n]/(Y^p - F)$$

as a subscheme of the weighted projective space

$$\mathbf{P}_{\mathbb{Z}} = \mathbb{P}_{\mathbb{Z}}(m, 1, \ldots, 1) = \operatorname{Proj} \mathbb{Z}[Y, X_0, \ldots, X_n].$$

There is a morphism

$$Z_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z},$$

whose special fiber over (p) is a degree p cyclic cover of projective n-space over $\mathbb{F}_p$ ramified over a divisor of degree mp, and whose generic fiber is similarly a cyclic cover of projective n-space over $\mathbb{Q}$. By choosing F to be sufficiently general, the general fiber is smooth (though the special fiber never is; see Exercise 4.9). Furthermore, a sufficiently general choice of F ensures that its reduction modulo p has only non-degenerate critical points in each affine patch, in other words that it satisfies Assumption 4.19. Therefore, by Theorem 4.4, the special fiber is not ruled. Now the idea is to apply the following theorem of Matsusaka stating that ruledness is well-behaved in families.

THEOREM 4.23 (Matsusaka's theorem (Mat68)). *Let V be a discrete valuation ring with quotient field K and residue field k. Let Z_S be a normal irreducible projective scheme over $S = \operatorname{Spec} V$.*

1. *If the generic fiber of the natural projection $Z_S \to S$ is ruled over K, then each irreducible component of the special fiber is ruled over k.*
2. *Assume in addition that the special fiber is reduced over $\bar{k}$. Then, if the generic fiber of the natural projection $Z_S \to S$ is ruled over $\bar{K}$, then each irreducible component of the special fiber is ruled over $\bar{k}$.*

CAUTION 4.24. The analog of Theorem 4.23 for rational instead of ruled varieties definitely fails. For instance, consider the morphism of schemes induced by the map

$$\mathbb{Z}_{(p)} \to \mathbb{Z}_{(p)}[Y, X_0, X_1, X_2]/\big(Y^3 - X_0^3 - X_1^3 - pX_2^3\big).$$

The general fiber is a smooth cubic surface, hence rational over $\bar{\mathbb{Q}}$. The special fiber is a cone over a smooth cubic curve E (as long as $p \neq 3$), which is birational to $E \times \mathbb{P}^1$. Thus the special fiber is ruled but not rational. It is not known whether similar examples exist in which the special fiber is smooth, but in any case, we wish to apply this result to a case where the special fiber is singular.

This example underscores the reason we are led to consider nonruled varieties in our quest for nonrational ones: ruledness is better behaved in families than rationality. We can not conclude that a special member of a family is rational when we know that the generic member is rational.

Matsusaka's theorem is proved in the next section. To keep things elementary, we prove only the following weak form of Matsusaka's theorem: If the general fiber is *rational,* then the components of the special fiber are ruled. This is sufficient to conclude the existence of nonrational cyclic covers. For the full proof of Theorem 4.23, the reader is referred to Kollár (1996, p. 184).

Before proving Theorem 4.23, we show how to use it to deduce the existence of nonrational Fano varieties in characteristic zero.

Proof that Theorem 4.23 implies Theorem 4.2. First we prove that the cyclic cover Z is nonrational for the most general choice of F.

The monomials in F are indexed by a set

$$\mathcal{I} = \left\{ (i_0, \ldots, i_n) : i_0 \geq 0,\ \sum_j i_j = mp \right\},$$

and F can be written as

$$F = \sum_{I \in \mathcal{I}} a_I X^I.$$

The most general case is when the a_I are algebraically independent over $\mathbb{Q}$. In this case, we might as well think of the a_I as variables, and consider the ring $R := \mathbb{Z}[a_I : I \in \mathcal{I}]$. We have the "universal cyclic cover" $\tilde{Z}$ mapping to $\mathbb{P}^n_R$, namely, the degree p cyclic cover of $\mathbb{P}^n_R$ ramified over the divisor defined by F.

The ring R is very big, so we localize at the ideal (p). Then $R_{(p)}$ is a discrete valuation ring whose residue field is the purely transcendental extension of $\mathbb{F}_p$ obtained by adjoining the a_I, and whose quotient field is likewise the analogous purely transcendental extension of $\mathbb{Q}$. We consider the natural map

$$\tilde{Z} \to \mathbb{P}^n_{R_{(p)}} \to \operatorname{Spec} R_{(p)}.$$

We now compare the special and the generic fibers of this map using Matsusaka's theorem.

Applying Theorem 4.23(2), we see that if the generic fiber is rational over the algebraic closure of $\mathbb{Q}(a_I : I \in \mathcal{I})$, then the special fiber is ruled over the algebraic closure of $\mathbb{F}_p(a_I : I \in \mathcal{I})$. However, this special fiber can not be ruled by Theorem 4.4. We conclude that the generic fiber is not geometrically rational. This implies that this cyclic cover can not be rational over any bigger field, including $\mathbb{C}$, by Proposition 3.33.

This completes the proof in the case where the a_I are algebraically independent over $\mathbb{Q}$. Finally, note that a set of elements of a field $\{b_I \in K : I \in \mathcal{I}\}$ is algebraically dependent over $\mathbb{Q}$ if and only if they satisfy an equation $H(b_I : I \in \mathcal{I}) = 0$ where H is a polynomial with integer coefficients. There are only countably many such equations H, so the set of all algebraically independent $|\mathcal{I}|$-tuples

$$\{b_{\mathcal{I}} = (b_I : I \in \mathcal{I})\} \subset K^{|\mathcal{I}|}$$

is "very general" in the sense defined immediately following Theorem 4.2. Thus we have shown that a cyclic cover of degree p ramified over a very general hypersurface of degree mp is never geometrically rational.

More precisely, the proof works for a polynomial F if and only if after some coordinate change its reduction to characteristic p has nondegenerate critical points. In some cases this can be checked easily but in general this is a rather subtle condition.

Finally, by choosing m, n and p to be within the stated range of Theorem 4.2, we can also assure that the cyclic cover Z is a smooth Fano variety in characteristic zero, completing the proof that Theorem 4.2 follows from its characteristic p analog. $\square$

4.5 Matsusaka's theorem and Abhyankar's lemma

We now complete the proof of the existence of nonrational Fano varieties by proving the weak form of Matsusaka's theorem concluding that the special fibers are not rational. The proof of the general version of Theorem 4.23 differs only in some technical points, we refer to Kollár (1996, p. 184) for details.

We assume that V is a localization of a finitely generated algebra over a field or over the integers. This is the only case that we used in the proof of Theorem 4.2. The general case can be reduced to this by showing that Z is defined over a finitely generated subring of V.

Proof of Theorem 4.23 in the weak form. Let $\operatorname{Spec} K$ be the generic point of S and $\operatorname{Spec} k$ be the closed point of S. Assume that the generic fiber Z_K is rational over K. Since S is birationally equivalent to $\operatorname{Spec} K$, a birational map

$$\mathbb{P}^n_S \times_S \operatorname{Spec} K = \mathbb{P}^n_K \dashrightarrow Z_K = Z_S \times_S \operatorname{Spec} K$$

defines a birational map

$$\phi : \mathbb{P}^n_S \dashrightarrow Z_S$$

over S. (If the generic fiber is only ruled, we can find a S-scheme $\mathbb{P}^1_S \times_S W$ mapping birationally onto Z_S over S. The following proof works only if this scheme is both regular and proper over S.)

Let Γ_S be the normalization of the closure of the graph (in $\mathbb{P}^n_S \times_S Z_S$) of the birational map ϕ. The normalization is a finite, birational morphism. So composing with the natural projections, we have proper birational morphisms $\pi_1 : \Gamma_S \to Z_S$ and $\pi_2 : \Gamma_S \to \mathbb{P}^n_S$.

Consider the special fiber Z_k of $Z_S \to S$. Being defined by a single equation, namely the pull-back of the uniformizing parameter u, all components of Z_k have codimension one. Likewise, the components of the special fiber Γ_k are all codimension one. Now, because $\pi_1 : \Gamma_S \to Z_S$ is a proper birational map of normal schemes, it is an isomorphism in codimension one (on the base). This means that for each irreducible component Z'_k of Z_k, there corresponds a unique irreducible component Γ'_k of Γ_k mapping birationally to it. Therefore, it suffices to show that the reduced irreducible divisor Γ'_k is ruled.

Consider the restriction of π_2 to Γ'_k. This gives a morphism $\Gamma'_k \to \mathbb{P}^n_k$, where $\mathbb{P}^n_k$ is the special fiber of $\mathbb{P}^n_S \to S$. If $\Gamma'_k \to \mathbb{P}^n_k$ is birational, the proof of Theorem 4.23(1) is complete: then Γ'_k, and hence the birationally equivalent variety Z_k, is birationally equivalent to $\mathbb{P}^n_k$.

Otherwise, Γ'_k is an exceptional divisor of the proper birational morphism $\pi_2 : \Gamma_S \to \mathbb{P}^n_S$. Exceptional divisors of proper birational maps to regular schemes are always ruled; this is an old result of Abhyankar that we present below as Theorem 4.26. Thus the proof of (1) is complete after Theorem 4.26 is proved.

Next consider (2). The scheme Z_K is rational over $\bar{K}$, so it is rational already over a finite degree subextension $K \subset K' \subset \bar{K}$. The normalization of V in K' has finitely many maximal ideals, let V' be its localization at one of these primes. Let k' be the residue field of V' and $S' := \operatorname{Spec} V'$. Set $Z' := Z \times_S S'$ and let $n : Z'' \to Z'$ be the normalization.

We can apply (1) to Z'' to conclude that every irreducible component of $Z''_{k'}$ is ruled over k'. Thus we are done if the normalization map induces birational maps between the irreducible components of $Z''_{k'}$ and the irreducible components of $Z'_{k'}$. The latter holds if Z' is normal at the generic point of each component of the special fiber. The closed fiber is defined by a single equation (namely the pull-back of the uniformizing parameter of V'), and we are assuming that Z_k and hence $Z'_{k'} = Z_{k'}$ are geometrically reduced. It follows easily from Exercise 4.25 that Z' is regular at the generic point of each component of the special fiber. Thus (1) implies (2). $\square$

EXERCISE 4.25. Let R be a local ring. Suppose that there exists a nonzero divisor $u \in m_R$ such that R/u is regular. Prove that R is regular.

THEOREM 4.26 (Abhyankar's lemma). *Let $\pi : Y \to X$ be a proper birational morphism of irreducible schemes, with Y normal and X regular. Then every exceptional divisor of π is ruled over its image. That is, if E is an integral subscheme of Y of codimension one, whose image E' has codimension greater than one in X, then E is birationally equivalent over E' to a scheme $W \times_{E'} \mathbb{P}^1_{E'}$.*

We first point out that when Y and X are algebraic varieties defined over a field k of characteristic zero, Theorem 4.26 follows easily from Hironaka's theorem on resolution of birational maps (although Abhyankar's 1956 proof came first). In this case, the morphism π^{-1} factors through a sequence of blowups along smooth centers:

$$\begin{tikzcd} & & & Y \arrow[d, "\pi"] \\ X_r \arrow[r, "\sigma_r"'] \arrow[urrr, "f"] & \cdots \arrow[r, "\sigma_2"'] & X_1 \arrow[r, "\sigma_1"'] & X \end{tikzcd}$$

Here, each $\sigma_{i+1} : X_{i+1} \to X_i$ is a blowing up along a nonsingular center, and $f : X_r \to Y$ is a birational morphism from the nonsingular variety X_r. To see that E is ruled, there is no harm in replacing Y by X_r and E by its birational transform on X_r. Because E is exceptional for the composition of the blowups $X_r \to X$, its image on some X_i must be an exceptional divisor for some blowup σ_i. The exceptional divisor of a blowup of a nonsingular variety along a nonsingular center is a projective space bundle over the center; in particular, such exceptional divisors, including E, must be ruled.

REMARK 4.27. Because we are interested only in birational properties, it is not actually necessary to use the full strength of Hironaka's deep theorem. The idea of the above argument can be adapted as follows. We construct the tower of blowups $\sigma_i : X_i \to X_{i-1}$ by blowing up the image E_{i-1} of E on X_{i-1}, but always restricting to the nonsingular loci of the E_{i-1}, so that each X_i is regular. Again, the exceptional fiber of the blowup of a regular scheme along a regular subscheme is a projective space bundle over the center of the blowup. The idea is to repeat this process until eventually the image of E on some X_i is a divisor; in this case E is ruled because it is birationally equivalent to its image E_i. The only problem is that the process of blowups may go on forever: it is not clear that the center of image of E ever becomes a divisor on X_i. Of course, the image E_i is a divisor on X_i if and only if the next blowup σ_{i+1} is an isomorphism. Hence the key point is to prove that: *The above sequence of blowups becomes trivial after finitely many steps.*

We prove this key point in §4.29 by keeping track of a numerical invariant that drops with each nontrivial blowup. Abhyankar's original proof uses valuation theory; see Kollár and Mori (1998, 2.45) for a general argument along these lines, which works even for X singular. Rather than reproduce this proof here in full generality, we show a nice geometric method in the case where X is regular and Y is of finite type over X. This is certainly sufficient for our purposes.

The numerical invariant we use is the order of vanishing along E of the pull back of a local generator of ω_{X_i} to ω_Y, provided that one can make sense of the sheaves ω_{X_i} and ω_Y and that one can define a pull-back map $\pi^*\omega_{X_i} \to \omega_Y$. For example, when Y and X are of finite type over an algebraically closed field, there is no problem making sense of ω_X and ω_Y and the argument works in this case.

In carrying out this argument, complications arise when the schemes X and Y are not defined over some base field. When X and Y are both smooth over some base scheme S, one can try to work with the relative canonical modules $\omega_{X/S}$. This sometimes works (for instance, if E is flat over S), but unfortunately, it breaks down precisely in the case we need it. The trouble arises because we must work with *regular* schemes that *may not be smooth* over the base scheme. Indeed, the following situation is typical: the scheme X may be $\mathbb{A}^n_S$ with $S = \operatorname{Spec} \mathbb{Z}_p$. We blow up a regular subscheme E_0 which maps to the closed point of S, say the closed point defined by $(p, x_1, \ldots, x_n)$. The resulting blowup scheme X_1 is regular, but it is not smooth over S. In this case, it is hard to define a relative canonical module $\omega_{X_1/S}$ that has the properties needed to carry out the argument along the lines suggested above.

However, we will be able to adapt the idea by working with the relative canonical modules for the birational maps $X_i \to X_0$. In fact, the duals of these canonical modules, the so-called Jacobian ideals, are more convenient to work with. This idea is from Johnston (Joh89).

4.6 Relative Canonical modules and Jacobian ideals

Let Y be a scheme of finite type over X, and suppose that $Y \to X$ has relative dimension d. We say that Y is smooth over X if the sheaf of relative Kähler differentials $\Omega_{Y/X}$ is a locally free $\mathcal{O}_Y$ module of rank d. In this case, the *relative canonical module* $\omega_{Y/X}$ is defined to be the invertible sheaf $\bigwedge^d \Omega_{Y/X}$.

When Y is normal and smooth in codimension one over X, the relative canonical module $\omega_{Y/X}$ can be defined as the unique reflexive $\mathcal{O}_Y$ module that agrees with the above construction on the smooth locus of $Y \to X$. Equivalently, $\omega_{Y/X}$ is the double dual of the $\mathcal{O}_Y$ module $\bigwedge^d \Omega_{Y/X}$. Although this canonical module is not necessarily invertible, it still can be interpreted as the "determinant" of $\Omega_{Y/X}$ via the natural map

$$\wedge^d \Omega_{Y/X} \to \omega_{Y/X} = (\wedge^d \Omega_{Y/X})^{**}$$

which is neither injective nor surjective in general.

This method of defining the canonical module fails when Y is not smooth in codimension one over X (for example, when $Y \to X$ is a blowup). Let us compute the "determinant" of $\Omega_{Y/X}$ in a different way so that it generalizes to this case. Fix an embedding $i : Y \hookrightarrow W$ where W is smooth over X, for instance, W may be taken to be an open subset of affine space over X. Consider the conormal complex:

$$\mathcal{I}_Y/\mathcal{I}_Y^2 \xrightarrow{d} i^*\Omega_{W/X} \to \Omega_{Y/X} \to 0$$

where $\mathcal{I}_Y \subset \mathcal{O}_W$ is the ideal sheaf of Y in W. If $\mathcal{I}_Y$ is locally generated by a regular sequence, as it must be, for instance, when both X and Y are regular, then the conormal sequence is exact also on the left:

$$0 \to \mathcal{I}_Y/\mathcal{I}_Y^2 \xrightarrow{d} i^*\Omega_{W/X} \to \Omega_{Y/X} \to 0.$$

This suggests a method for computing the "determinant" of $\Omega_{Y/X}$ when X and Y are regular. In this case $\mathcal{I}_Y/\mathcal{I}_Y^2$ and $i^*\Omega_{W/X}$ are both locally free, hence we can propose the definition

$$\omega_{Y/X} := \wedge^n i^*\Omega_{W/X} \otimes \left(\wedge^{n-d} \left(\mathcal{I}_Y/\mathcal{I}_Y^2\right)\right)^{-1},$$

where n is the relative dimension of W over X and d is the relative dimension of Y over X. This agrees with our previous definition, but makes sense even when $Y \to X$ is not smooth in codimension one. The module $\omega_{Y/X}$ is invertible (provided Y is generically smooth over X so that ranks are as expected). We do not need to worry about the dualizing properties of the sheaf.

Now, in order to prove Theorem 4.26, we must consider the case where $Y \to X$ is a birational morphism of regular schemes. The relative dimension is zero. The map of rank n locally free $\mathcal{O}_Y$-modules

$$d : \mathcal{I}_Y/\mathcal{I}_Y^2 \hookrightarrow i^*\Omega_{W/X}$$

gives rise to a map of invertible $\mathcal{O}_Y$-modules

$$\bigwedge\nolimits^n \mathcal{I}_Y/\mathcal{I}_Y^2 \hookrightarrow \bigwedge\nolimits^n i^*\Omega_{W/X}.$$

Tensoring with $(\bigwedge^n i^*\Omega_{W/X})^{-1}$ we get an exact sequence

$$0 \to \bigwedge\nolimits^n \mathcal{I}_Y/\mathcal{I}_Y^2 \otimes (\bigwedge\nolimits^n i^*\Omega_{W/X})^{-1} \to \mathcal{O}_Y \to \mathcal{Q} \to 0.$$

Here, $\omega_{Y/X} = \bigwedge^n i^*\Omega_{W/X} \otimes (\bigwedge^n \mathcal{I}_Y/\mathcal{I}_Y^2)^{-1}$ so that its dual, $\omega_{Y/X}^{-1} = \bigwedge^n \mathcal{I}_Y/\mathcal{I}_Y^2 \otimes (\bigwedge^n i^*\Omega_{W/X})^{-1}$ is a sheaf of ideals in $\mathcal{O}_Y$. It is often called the *Jacobian ideal* and denoted by $\mathcal{J}_{Y/X}$. Note also that $\mathcal{Q}$ is some torsion $\mathcal{O}_Y$-module supported on the nonsmooth locus of $Y \to X$, so that the Jacobian ideal defines the nonsmooth locus of $Y \to X$.

To explain the name, choose local coordinates $x_1, \ldots, x_n$ for W over X, such that the dx_i are a free basis for $\Omega_{W/X}$. Suppose that $\mathcal{I}_Y$ is defined locally by the regular sequence $f_1, \dots, f_n$. Then the map of free $\mathcal{O}_Y$ modules

$$d : \mathcal{I}_Y/\mathcal{I}_Y^2 \hookrightarrow i^*\Omega_{W/X}$$

sends the class of a generator $\bar{f}_i$ to $\sum_{i=1}^n \frac{\partial f_i}{\partial x_j} dx_j$. In other words, the map is defined by the Jacobian matrix $(\frac{\partial f_i}{\partial x_j})$, so that

$$\bigwedge^n \mathcal{I}_Y/\mathcal{I}_Y^2 \hookrightarrow \bigwedge^n i^*\Omega_{W/X}$$

is defined by its determinant. In particular, the Jacobian ideal $\mathcal{J}_{Y/X}$ is locally generated by this Jacobian determinant.

The proof of Theorem 4.26 rests on the simple observation that Jacobian ideals are multiplicative.

EXERCISE 4.28. If $\pi : Z \to Y \to X$ are finite type birational maps of regular schemes, then

$$\mathcal{J}_{Z/X} = (\mathcal{J}_{Z/Y})(\mathcal{J}_{Y/X}\mathcal{O}_Z)$$

as ideals of $\mathcal{O}_Z$. Here $\mathcal{J}_{Y/X}\mathcal{O}_Z$ denotes the ideal of $\mathcal{O}_Z$ generated by the pullbacks of generators of $\mathcal{J}_{Y/X}$; because $\mathcal{J}_{Y/X}$ is invertible, this is the same as $\pi^*\mathcal{J}_{Y/X}$.

PROOF OF THEOREM 4.26 ASSUMING Y IS OF FINITE TYPE OVER X. Because Y is normal, it is regular in codimension one. So we are free to replace Y by an open set containing the generic point of E so as to assume that Y is regular.

The divisor E is exceptional for the given birational map $\pi : Y \to X$, so its image under π is a subscheme of codimension at least two. Let us denote this image subscheme by E_0. Note that E_0 is a reduced and irreducible closed subscheme of the regular scheme X.

The subscheme E_0 need not be regular. However, because it is reduced, the locus of its nonregular points is a proper closed subscheme. So we can replace X by an open set X_0 in which E_0 is regular. Let $\sigma_1 : X_1 \to X_0$ be the blowup of the regular scheme X_0 along the regular subscheme E_0. The resulting scheme X_1 is regular, and the exceptional fiber of the blowup is a projective space bundle over the center E_0 of the blowup. In particular, the exceptional divisor is ruled.

Let E_1 be the image of $E \subset Y$ in X_1 under the rational map $\sigma_1^{-1} \circ \pi : Y \dashrightarrow X_1$. Of course, E_1 must be contained in the exceptional set for σ_1, but it may be strictly smaller. If E_1 is codimension one in X_1, then E_1 must be this exceptional divisor. In this case, E is ruled and the proof is complete.

Otherwise, E_1 has codimension larger than one in X_1 and we repeat the process of replacing X_1 and E_1 by an open subset on which E_1 is regular and blowing up along E_1. In this way, we construct a sequence of blowups $\sigma_i : X_i \to X_{i-1}$. Each X_i is regular (but not necessarily smooth over the base scheme) and each exceptional fiber is ruled and contains the image E_i of E. The process terminates (meaning σ_{i+1} is an isomorphism) if and only if E_i is codimension one in X_i. If the process terminates, the proof is complete, because then E is birational to the exceptional divisor of some σ_i, and so E must be ruled.

4.29. TERMINATION OF THE PROCESS. If the process does not terminate, we have a sequence of blowings up of regular schemes

$$X_0 \xleftarrow{\sigma_1} X_1 \xleftarrow{\sigma_2} X_2 \xleftarrow{\sigma_3} X_3 \cdots$$

where no σ_i is an isomorphism (we say "σ_i is a nontrivial blowup"). Such a nontrivial blowup is never smooth (nor even flat!). Since the Jacobian ideal $\mathcal{J}_{X_i/X_{i-1}} \subset \mathcal{O}_{X_i}$ defines the nonsmooth locus of the blowup $X_i \to X_{i-1}$, none of these Jacobian ideals can be the unit ideal. Indeed, because the blowup $X_i \to X_{i-1}$ is not smooth along the exceptional divisor, the Jacobian ideal remains a proper ideal after localizing along any component of an exceptional divisor.

The rational map $\pi_i : Y \dashrightarrow X_i$ is a morphism on some open set $Y_i \subset Y$ containing the generic point of E. In particular, by Exercise 4.28, the morphisms

$$Y_i \to X_i \to X_0$$

induce a multiplicative relation of Jacobian ideals in $\mathcal{O}_{Y_i}$:

$$\mathcal{J}_{Y_i/X_0} = (\mathcal{J}_{Y_i/X_i})(\mathcal{J}_{X_i/X_0}\mathcal{O}_{Y_i}).$$

Localizing along E, we have a multiplicative relation of *proper* ideals

$$\mathcal{J}_{Y_i/X_0}\mathcal{O}_{Y,E} = (\mathcal{J}_{Y_i/X_i}\mathcal{O}_{Y,E})(\mathcal{J}_{X_i/X_0}\mathcal{O}_{Y,E})$$

in the discrete valuation ring $\mathcal{O}_{Y,E}$. In particular, the pullback of each Jacobian ideal $\mathcal{J}_{X_i/X_0}$ to $\mathcal{O}_{Y,E}$ strictly contains the fixed ideal $\mathcal{J}_{Y_i/X_0}\mathcal{O}_{Y,E}$. Since this latter ideal of the local ring $\mathcal{O}_{Y,E}$ depends only on a small neighborhood of E in Y, we denote it by $\mathcal{J}_{Y/X_0}$.

Likewise, using the multiplicative property for Jacobian ideals for the blowups $X_{i+1} \to X_i \to X_0$, we see that after pulling back to Y and localizing along E, the ideal $\mathcal{J}_{X_i/X_0}\mathcal{O}_{Y,E}$ strictly contains the ideal $\mathcal{J}_{X_{i+1}/X_0}\mathcal{O}_{Y,E}$. We are led to a sequence of inclusions

$$\mathcal{J}_Y \subset \cdots \subsetneq \mathcal{J}_{X_i} \subsetneq \mathcal{J}_{X_{i-1}} \subsetneq \cdots \subsetneq \mathcal{J}_{X_1} \subsetneq \mathcal{J}_{X_0}$$

in the discrete valuation ring $\mathcal{O}_{Y,E}$ (the notation for "relative to X_0" and "localize along E" has been suppressed).

This leads immediately to a contradiction: fixing a uniformizing parameter t for $\mathcal{O}_{Y,E}$, and setting $\mathcal{J}_Y = (t^m)$, it is obvious that at most m ideals can be properly contained between $\mathcal{J}_Y$ and $\mathcal{O}_{Y,E}$. The process must terminate after at most m blowups, and the proof is complete. □

REMARK 4.30. One can associate the numerical invariant given by the length of

$$\mathcal{J}_{X_i}/\mathcal{J}_Y$$

to each blowup. The proof showed that this number is strictly decreasing for a nontrivial blowup. This number can be interpreted as the "discrepancy along E" between differentials on Y and on X_i. In this sense, the proof we have given above is very close in spirit to the proof we suggested in the classical case. The difference is that the differentials here are relative to the scheme X_0, whereas in the classical case, the differentials are relative to the ground field.

4.7 Explicit examples (by J. Rosenberg)

In this section we write down some explicit polynomials with non-degenerate critical points, thus establishing concrete examples of nonruled Fano varieties over $\mathbb{Q}$.

PROPOSITION 4.31. *Given a prime p, and integers n and m with $n > 0$, $n \not\equiv -1 \mod p$, and $mp \geq 3$, let $F \in \mathbb{F}_p[x_0, \dots, x_n]$ be the homogeneous polynomial of degree mp*

$$F(x_0, \dots, x_n) = \sum_{i=0}^{n} x_i^{mp-1} x_{i+1},$$

where we understand subscripts to be taken mod $n+1$. Then any dehomogenization f of F

$$f(x_0, \dots, \hat{x}_i, \dots, x_n) = F(x_0, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n)$$

has only isolated critical points in $\bar{\mathbb{F}}_p$, and all of them are non-degenerate.

PROOF. From the cyclic symmetry of F, it is clear we need only consider the dehomogenization

$$f(x_1, \dots, x_n) = F(1, x_1, \dots, x_n).$$

Then we have

$$\frac{\partial F}{\partial x_i} = x_{i-1}^{mp-1} - x_i^{mp-2} x_{i+1},$$

$$\frac{\partial^2 F}{\partial x_i \partial x_{i+1}} = -x_i^{mp-2},$$

$$\frac{\partial^2 F}{\partial x_i^2} = 2x_i^{mp-3} x_{i+1},$$

and all other second partials of F are zero. We find that any critical point of f satisfies

$$\begin{aligned} 1 - x_1^{mp-2} x_2 &= 0, \\ x_1^{mp-1} - x_2^{mp-2} x_3 &= 0, \\ x_2^{mp-1} - x_3^{mp-2} x_4 &= 0, \\ &\vdots \\ x_{n-2}^{mp-1} - x_{n-1}^{mp-2} x_n &= 0, \\ x_{n-1}^{mp-1} - x_n^{mp-2} &= 0. \end{aligned}$$

We conclude that the critical points of f are exactly those points with

$$x_i = \zeta^{\left(\sum_{j=0}^{i-1} (1-mp)^j\right)}$$

for some ζ with

$$\zeta^{\left(\sum_{j=0}^{n} (1-mp)^j\right)} = 1.$$

In particular, all critical points are isolated and all their coordinates are nonzero.

To see that these points are non-degenerate, we write down the Hessian of f,

$$H = \begin{pmatrix} 2x_1^{mp-3}x_2 & -x_1^{mp-2} & & & & \\ -x_1^{mp-2} & 2x_2^{mp-3}x_3 & -x_2^{mp-2} & & & \\ & -x_2^{mp-2} & 2x_3^{mp-3}x_4 & & & \\ & & & \ddots & & \\ & & & & 2x_{n-1}^{mp-3}x_n & -x_{n-1}^{mp-2} \\ & & & & -x_{n-1}^{mp-2} & 2x_n^{mp-3} \end{pmatrix}$$

and compute its determinant at each of the critical points. If we let H_j be the upper left $j \times j$ submatrix of H, and $h_j = \det(H_j)$, we see that we have a recursion for $\det(H) = h_n$:

$$\begin{aligned} h_0 &= 1, \\ h_1 &= 2x_1^{mp-3}x_2, \\ h_j &= 2x_j^{mp-3}x_{j+1}h_{j-1} - x_{j-1}^{2mp-4}h_{j-2}, \text{ for } 1 < j \le n, \end{aligned}$$

where we understand x_{n+1} to equal 1. We show that modulo the ideal of first partials $(\frac{\partial f}{\partial x_i})$, this reduces to

$$h_j \equiv (j+1)\prod_{i=1}^{j} x_i^{mp-3}x_{i+1}.$$

Clearly this is true for $j = 0$ and $j = 1$. Now, inductively, for $1 < j \le n$,

$$\begin{aligned} h_j &= 2x_j^{mp-3}x_{j+1}h_{j-1} - x_{j-1}^{2mp-4}h_{j-2} \\ &\equiv 2x_j^{mp-3}x_{j+1}h_{j-1} - x_{j-1}^{mp-3}x_j^{mp-2}x_{j+1}h_{j-2} \\ &\equiv 2jx_j^{mp-3}x_{j+1}\prod_{i=1}^{j-1}x_i^{mp-3}x_{i+1} \\ &\quad - (j-1)x_{j-1}^{mp-3}x_jx_j^{mp-3}x_{j+1}\prod_{i=1}^{j-2}x_i^{mp-3}x_{i+1} \\ &= (j+1)\prod_{i=1}^{j}x_i^{mp-3}x_{i+1}, \end{aligned}$$

as desired. In particular, $\det(H) = h_n$ is equal to $n+1$ times a monomial, which is nonzero for any critical point of f. So f, and similarly any dehomogenization of F, has no degenerate critical points. $\square$

This produces a host of explicit examples of nonrational smooth projective Fano varieties of every dimension. For example:

COROLLARY 4.32. *Let $G(X_0, \ldots, X_4)$ be any homogeneous polynomial of degree 6 with integral coefficients. Then the double cover of $\mathbb{P}^4$ ramified over the divisor defined by*

$$\sum_{i=0}^{4} X_i^5 X_{i+1} - 2G(X_0, \ldots, X_4)$$

is a smooth, nonrational Fano variety over $\mathbb{Q}$.

PROOF. $\sum_{i=0}^{4} X_i^5 X_{i+1}$ has non-degenerate critical points over a field of characteristic 2, hence the corresponding double cover is not ruled by (4.4). Thus the corresponding variety over $\mathbb{Q}$ is not rational by (4.23).

By an explicit computer calculation, none of the critical points are on the hypersurface $(\sum_{i=0}^{4} X_i^5 X_{i+1} = 0)$. Thus the hypersurface

$$\left(\sum_{i=0}^{4} X_i^5 X_{i+1} + 2G(X_0, \ldots, X_4) = 0\right) \subset \mathbb{P}^4$$

is smooth over $\mathbb{Q}$, since its reduction to characteristic 2 is smooth. Therefore the double cover is also smooth by (4.1). $\square$

5

The Noether–Fano method for proving nonrationality

In this chapter, we present the higher-dimensional version of the Noether–Fano technique introduced in Chapter 2 to treat cubic surfaces. The Noether–Fano method forms the basis of most current approaches to birational problems for nearly rational varieties of higher dimension. We apply the method in two cases. First, in Section 3, we show that certain Fano hypersurfaces in weighted projective spaces are not rational. These are by far the simplest cases known but they were discovered only recently. Then, in Section 4, we turn to the very first application, proving that no smooth quartic threefold is rational. This is quite a bit harder and the proof of a key part is completed only in Chapter 6.

The Noether–Fano method is based on the idea that a Fano variety birationally equivalent to another in a nontrivial way must admit a "very singular" linear system. In Chapter 2, we were concerned with surfaces and "very singular" was a simple multiplicity statement. In higher dimensions, the meaning of "very singular" is more subtle and indeed, we expend considerable energy in Chapter 6 to prove a particular numerical bound on the singularities of such a system.

In Section 1, we outline the method, beginning with the Noether–Fano inequality, Theorem 5.5. The main point is that a birational map between Fano varieties of Picard number one arises from a linear system that admits a *maximal center*, a subvariety of its base locus along which it is particularly singular.

In Section 2, some numerical consequences of the existence of maximal centers are discussed. The main results are Proposition 5.11, giving a simple and quite general multiplicity bound on a linear system at a maximal center, and Theorem 5.20, giving a much more subtle numerical consequence of a zero-dimensional maximal center on a threefold. Proposition 5.11, which is sufficient to deal with maximal centers on surfaces, is proved here. However, Theorem 5.20 is difficult and its proof is postponed until Chapter 6.

In Section 3, we apply the Noether–Fano method to show that no smooth member of a certain family of Fano hypersurfaces in weighted projective space is rational. In a sense, this is a more general result than what was proved in Chapter 4, where our methods were valid only for "very general" members of the corresponding families. In fact, we show that our hypersurfaces are *rigid* Fano varieties: every birational self-map is an automorphism. According to the Noether–Fano method, to prove these statements we must disprove the existence of maximal centers for certain kinds of linear systems on our hypersurfaces. We are able to do this easily using the elementary numerical consequence of maximal centers, Proposition 5.11 – at least after a thorough study of the geometry of these hypersurfaces.

In Section 4 we prove that no smooth quartic in projective four-space is rational. Again, this is straightforward application of the Noether–Fano method. However, our proof does depend on the deeper numerical result, Theorem 5.20. Although one approach to proving Theorem 5.20 is discussed in Section 2, a self-contained proof of this result appears only later in Chapter 6.

5.1 The Noether–Fano method

In this section, we introduce the Noether–Fano method for studying birational maps between projective varieties. The main point is that a nontrivial birational map between sufficiently nice varieties is given by a linear system admitting a *maximal center* (Definition 5.9). Very roughly, a maximal center is a place where there is a significant discrepancy between the pull-back and the birational transform (under some map) of the linear system. We begin by reviewing the notion of birational transform for a linear system.

5.1. Birational Transforms of Linear Systems. Let $g : X \dashrightarrow Y$ be a birational map between normal varieties. Given a finite dimensional linear system H of Weil divisors X, we now define the *birational transform* g_*H of H on Y.

Let us start with a linear system M which has no fixed components. Such linear systems are also called *mobile*.

Let X^0 be an open subset of X such that g defines an isomorphism of X^0 with an open subset Y^0 of Y. Then $g|_{X^0}(M|_{X^0})$ is a mobile linear system on Y^0, and it uniquely extends to a mobile linear system on Y. This is the birational transform g_*M of M.

It is helpful to interpret birational transforms of linear systems in terms of the maps to projective space they determine. Let M be a mobile linear system on a normal variety X and let

$$\phi_M : X \dashrightarrow \mathbb{P}^n$$

be the corresponding rational map. If $g : X \dashrightarrow Y$ is a birational map from a normal variety, then the birational transform of a mobile linear system M on Y is the unique mobile linear system on Y corresponding to the composition map $\phi_M \circ g^{-1} : Y \dashrightarrow X \dashrightarrow \mathbb{P}^n$.

If H is an arbitrary linear system on X, then we can decompose H as $M + F$ where M and F are the mobile and fixed parts of H respectively. The birational transform of H is then

$$g_* H = g_* M + g_* F,$$

where $g_* F$ is the birational transform of the divisor F as defined in the Introduction.

Often we are given a birational map $f : Y \dashrightarrow X$ between normal varieties and a linear system H on X whose birational transform on Y is of interest. This leads to the odd looking notation $f_*^{-1} H$, meaning simply $g_* H$ where $g = f^{-1}$. In particular, if M is mobile and Cartier and f is a morphism, then $f_*^{-1} M$ is the mobile part of the linear system $f^* M$ on Y.

Some caution is in order when thinking about the members of a birationally transformed linear system. If g is a *morphism*, then the members of $g_* H$ are simply the birational transforms of the members of H. But if g has a nontrivial locus of indeterminacy, then this may not be the case for members of H that intersect it. It is true, however, that the *general* member of $g_* H$ is the birational transform of the general member of H. (For example, consider the case where H consists of lines in the plane $\mathbb{P}^2$ and $g : \mathbb{P}^2 \dashrightarrow X$ is a blowing up map at a point P. The birational transforms of the members of H are not even linearly equivalent to each other. The birational transform L' of a line through P is not a member of $g_* H$, although it does extend to the member $L' + E$.)

5.2. The Noether–Fano inequalities. Consider a linear system H of divisors on a smooth variety X. Given a birational morphism $p : Z \to X$ from a normal variety Z, we write

$$K_Z = p^* K_X + E_p \qquad \text{and} \qquad p_*^{-1} H = p^* H - F_p, \tag{5.2.1}$$

where both E_p and F_p are uniquely defined effective p-exceptional divisors. Indeed, although K_X and K_Z denote divisor classes, there is a unique member of $K_Z - p^* K_X$ supported only on the exceptional set, and this is E_p. Similarly, the members of the linear system $p^* H$ consist of members of $p_*^{-1} H$ plus a certain unique fixed component, namely F_p. (To check these statements, one must verify that no nontrivial combination of exceptional divisors is linearly trivial; we leave this to the reader.)

Our point of view will be that the relative sizes of E_p and F_p provide a measure of the complexity of the base locus of the linear system H. For the most naive example, note that if H is base point free, then F_p is trivial for any birational morphism p; conversely, if there exists p such that $p_*^{-1}H$ is base point free and F_p is trivial, then H is base point free. Exercise 5.4 below gives a more subtle illustration of how the relative size of E_p and F_p measures the singularities of the base locus of H.

REMARK 5.3. The smoothness assumption on X in the preceding discussion is not really necessary. The point is that on a smooth variety, every divisor is Cartier, so that the pull-backs f^*K_X and f^*H always make sense. If X is normal, then K_X still makes sense as a Weil divisor class: it is the closure of the canonical class on the smooth locus. (By the normality assumption, the nonsmooth locus has codimension at least two and so divisors on the smooth locus extend uniquely to Weil divisors on all of X.) However, the pull-backs p^*K_X and f^*H can be defined only if K_X and H are Cartier or at least $\mathbb{Q}$-Cartier. See Discussion 6.1.

EXERCISE 5.4. Let H be a mobile linear system on a smooth surface S. Let $g : S' \to S$ be any birational morphism from a normal variety S'. As in (5.2.1), write $K_{S'} = g^*K_S + E_g$ and $g_*^{-1}H = g^*H - F_g$. Show that the following are equivalent for each natural number m.

1. The $\mathbb{Q}$-divisor $E_g - \frac{1}{m}F_g$ is effective for every g.
2. The multiplicity of H at P is at most m, for every $P \in S$.

For a linear system H contained in $|-mK_S|$, condition (2) of Exercise 5.4 is familiar from the proofs of the theorems of Segre and Manin. The multiplicity version does not generalize to higher dimensions, but the equivalent form (1) does. The following is the content of the Noether–Fano inequality.

THEOREM 5.5. *Let $\phi : X \dashrightarrow X'$ be a birational map between smooth Fano varieties of Picard number one. Suppose that H' is a base point free linear system on X' contained in the complete linear system $|-m'K_{X'}|$. Let H be its birational transform on X, and suppose that H is contained in $|-mK_X|$. Then*

1. *$m \geq m'$;*
2. *if $m = m'$, then ϕ is an isomorphism;*

3. *fix a birational factorization*

$$\begin{tikzcd} & Z \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \arrow[rr, dashed, "\phi"] & & X' \end{tikzcd}$$

where Z is normal, and let E_p and F_p denote the p-exceptional divisors as defined in §5.2; if $E_p - \frac{1}{m}F_p$ is effective, then ϕ is an isomorphism.

Because of the appearance of the $\mathbb{Q}$-divisor $E_p - \frac{1}{m}F_p$, it is natural to use numerical equivalence in the proof. Two Cartier divisors D_1, D_2 on a projective variety are called *numerically equivalent* if $D_1 \cdot C = D_2 \cdot C$ for every irreducible curve $C \subset X$ where $D \cdot C$ denotes the intersection number of the divisor D with the curve C. Numerical equivalence is denoted by $D_1 \equiv D_2$. One can naturally extend numerical equivalence to $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors.

In the present case, all numerical equivalences become linear equivalences upon clearing denominators.

PROOF. Let H_Z denote the birational transform of H' (and so of H) on Z. As in (5.2.1) above, we write

$$K_Z = q^*K_{X'} + E_q, \qquad H_Z = q^*H' - F_q,$$

where E_q and F_q are effective q-exceptional divisors, and also

$$K_Z = p^*K_X + E_p, \qquad H_Z = p^*H - F_p,$$

where E_p and F_p are effective p-exceptional divisors. Let c be a rational number to be specified later. Then

$$\begin{aligned} K_Z + cH_Z &\equiv q^*(K_{X'} + cH') + E_q \quad \text{and} \\ K_Z + cH_Z &\equiv p^*(K_X + cH) + E_p - cF_p. \end{aligned} \tag{5.5.4}$$

In the first equivalence, we have used the fact that H' is base point free, which implies that $F_q = 0$.

Setting $c = \frac{1}{m'}$, we see that

$$K_Z + \tfrac{1}{m'}H_Z \equiv q^*(K_{X'} + \tfrac{1}{m'}H') + E_q \equiv E_q,$$

which is effective. So pushing down to X, we see that

$$K_X + \tfrac{1}{m'}H \equiv p_*(K_Z + \tfrac{1}{m'}H_Z) \equiv p_*(E_q)$$

is effective as well, and thus the divisor $m'K_X + H$ is equivalent to an effective divisor. Because H is linearly equivalent to $-mK_X$, we conclude that $(m' - m)K_X$ is effective. As $-K_X$ is ample, it follows that $m \geq m'$, and (1) is proved.

To prove (2), suppose that $m = m'$. Then

$$p_*(E_q) \equiv K_X + \tfrac{1}{m'}H \equiv K_X + \tfrac{1}{m}H = 0.$$

This means that every divisor in the support of E_q is exceptional for p. On the other hand, since the support of E_q is the whole q-exceptional set, we obtain that every q-exceptional divisor is also p-exceptional, that is,

$$\{q\text{-exceptional divisors}\} \subset \{p\text{-exceptional divisors}\}. \tag{5.5.5}$$

Because the Picard number of Z is equal to the Picard number of X plus the number of p-exceptional divisors (and likewise for X'), we also know that the number of q-exceptional divisors on Z is equal to the number of p-exceptional divisors. Combined with (5.5.5), this means that the exceptional sets, $\operatorname{Ex} p$ and $\operatorname{Ex} q$, are exactly the same. It follows that ϕ restricts to an isomorphism

$$\phi : X \setminus p(\operatorname{Ex} p) \xrightarrow{\cong} X' \setminus q(\operatorname{Ex} q).$$

Since ϕ is an isomorphism in codimension one, finally, Exercise 5.6 implies that ϕ is an isomorphism, and (2) is proved.

To prove (3), we set $c = \frac{1}{m}$ in formula (5.5.4). This gives

$$K_Z + \tfrac{1}{m}H_Z \equiv p^*(K_X + \tfrac{1}{m}H) + E_p - \tfrac{1}{m}F_p \equiv E_p - \tfrac{1}{m}F_p.$$

Pushing down to X' we get

$$K_{X'} + \tfrac{1}{m}H' \equiv q_*(K_Z + \tfrac{1}{m}H_Z) \equiv q_*(E_p - \tfrac{1}{m}F_p). \tag{5.5.6}$$

So if $E_p - \frac{1}{m}F_p$ is effective, then $K_{X'} + \frac{1}{m}H'$ is effective as well. As before, it follows that $m' \geq m$. Combining with (1), it follows that $m = m'$, and so ϕ is an isomorphism by (2). $\square$

Exercise 5.6 (Matsusaka and Mumford, 1964). Consider a birational map $\phi : X \dashrightarrow X'$ between smooth projective varieties. Assume that

1. there are codimension two subsets $W \subset X$ and $W' \subset X'$ such that ϕ restricts to an isomorphism from $X \setminus W$ to $X' \setminus W'$, and
2. there is an ample divisor on X whose birational transform on X' is also ample.

Show that ϕ is an isomorphism. (This exercise indicates that codimension two surgery operations are very constrained in algebraic geometry.)

EXAMPLE 5.7. To understand the Noether–Fano inequalities, let us revisit the surface case to see how they might be of use in proving the theorem of Segre from Chapter 2. Suppose that X is a cubic surface of Picard number one and that X' is the projective plane. Consider the hyperplane linear system H' on $\mathbb{P}^2$, and suppose that its birational transform on X is contained in $|-mK_X|$. Now Theorem 5.5 guarantees that if ϕ is not an isomorphism, then $E_p - \frac{1}{m}F_p$ is not effective. Hence, by Exercise 5.4, the linear system H has a base point of multiplicity greater than m. The first step in the proof of the theorem of Segre established precisely this; see the proof of Theorem 2.13.

In applications of the Noether–Fano inequalities, it is useful to have a geometric interpretation for the condition that $E_p - \frac{1}{m}F_p$ be effective for a given linear system H. In practice we know very little about Z, thus we are led to ask the question:

When is $E_p - \frac{1}{m}F_p$ effective for every $p : Z \to X$?

In a few cases, this can be decided by *ad hoc* methods, but it is better to set up some general machinery.

5.8. MAXIMAL CENTERS. By a *divisor over a variety X*, we mean any prime divisor E of Y, where Y is any normal variety admitting a birational (but not necessarily proper) morphism f to X. The *center* of a divisor E over X is its image $f_*(E)$ on X, that is, the closure of the set-theoretic image of E in X. A divisor is said to be *exceptional over X* if its center on X has codimension at least two. Clearly, if $Y \to X$ and $Y' \to X$ are two birational morphisms and the birational transform D' on Y' of some divisor D on Y is a divisor, then the centers of D and of D' on X are the same.

DEFINITION 5.9. Consider a mobile linear system H on a smooth variety X and a positive rational number c. Then a *maximal center* of the pair (X, cH) is the center on X of any divisor appearing with negative coefficients in the divisor $E_p - cF_p$, where E_p and F_p are as defined in §5.2 for some choice of birational morphism p. For historical reasons, the $\mathbb{Q}$-divisor $E_p - cF_p$ is frequently written in the form

$$(K_Y + p_*^{-1}(cH)) - p^*(K_X + cH).$$

Note that every maximal center of (X, cH) is contained in every member of H; in other words, every maximal center is contained in the base locus of H. The name maximal center is traditional; from the point of view of the general theory in Chapter 6, the name *non-canonical center* is perhaps more appropriate.

With the terminology of maximal centers, we can rephrase the main consequence of Theorem 5.5 as follows.

COROLLARY 5.10. *Let X be a smooth Fano variety of Picard number one. Suppose that there are no mobile linear systems H contained in the complete linear system $|-mK_X|$ with the property that $(X, \frac{1}{m}H)$ has a maximal center. Then every birational map from X to another smooth Fano variety of Picard number one is an isomorphism.*

For Corollary 5.10 to be useful, we need to find a way to find maximal centers or to disprove their existence. This is a rather difficult problem in general, with many questions wide open. In the next section, we discuss some numerical consequences of the existence of maximal centers in certain cases. These will be sufficient to show that certain families of Fano hypersurfaces in weighted projective space are not rational in Section 3 and that quartic threefolds are not rational in Section 4.

5.2 Numerical consequences of maximal centers

The Noether–Fano inequality, Theorem 5.5, shows the importance of understanding maximal centers of linear systems. The aim of this section is to derive numerical properties for linear systems with maximal centers. We are somewhat lucky that for quartic threefolds, and in a few other cases, these numerical properties alone imply that there are no "bad" maximal centers.

We begin with a simple multiplicity estimate for maximal centers that holds quite generally, as shown by the following generalization of Exercise 5.4.

PROPOSITION 5.11. *Let H be a mobile linear system on a smooth variety X, and let $W \subset X$ be a maximal center of $(X, \frac{1}{m}H)$. Then the multiplicity of H at each point of W is greater than m.*

To prove Proposition 5.11, assume to the contrary that the multiplicity of H is less than m at some point P in W. By the upper-semicontinuity of multiplicity, there is an open subset X^0 of X containing P such that $\mathrm{mult}_x\, H \leq m$ for every $x \in X^0$. On the other hand, $W \cap X^0$ is a maximal center for the restriction $H|_{X^0}$. Thus, after replacing X by X^0, Proposition 5.11 is implied by the following variant.

Proposition 5.12. *Let H be a mobile linear system on a smooth variety X such that the multiplicity of H is bounded above by m at each point of X. Then $(X, \frac{1}{m}H)$ does not have any maximal centers.*

Proof. The proof is more transparent if we assume the strong form of resolution of indeterminacies. That is, given any birational morphism $p : Z \to X$, we consider a diagram

$$\begin{tikzcd} & & & Z \arrow[d, "p"] \\ X_m \arrow[r, "f_m"'] \arrow[urrr, "h"] & \cdots \arrow[r, "f_2"'] & X_1 \arrow[r, "f_1"'] & X_0 = X \end{tikzcd}$$

where each $f_{i+1} : X_{i+1} \to X_i$ is a blowing up along a smooth center $V_i \subset X_i$ and $h : X_m \to Z$ is a birational morphism from the smooth variety X_m. Set

$$g_i = f_1 \circ \cdots \circ f_i : X_i \to X,$$

and let H_i denote the birational transform of H on X_i. Our aim is to prove by induction that

$$K_{X_i} + \tfrac{1}{m}H_i \equiv g_i^*(K_X + \tfrac{1}{m}H) + (\text{effective divisor}). \qquad (5.11.1_i)$$

After this is accomplished, the proof is complete, since the formula $(5.11.1_m)$ can be pushed down to Z to yield

$$K_Z + \tfrac{1}{m}H_Z \equiv p^*(K_X + \tfrac{1}{m}H) + (\text{effective divisor}),$$

where H_Z is the birational transform of H on Z.

To carry out the inductive step, fix one i and consider the blowup $f_{i+1} : X_{i+1} \to X_i$ of the smooth codimension c_i subvariety V_i of X_i. We know that

$$\begin{aligned} K_{X_{i+1}} &= f_{i+1}^* K_{X_i} + (c_i - 1)E_{i+1} \quad \text{and that} \\ f_{i+1}^* H_i &= H_{i+1} + (\operatorname{mult}_{V_i} H_i) \cdot E_{i+1}, \end{aligned}$$

where E_{i+1} denotes the exceptional divisor of f_{i+1}. Taking a suitable linear combination we have

$$K_{X_{i+1}} + \tfrac{1}{m}H_{i+1} \equiv f_{i+1}^*(K_{X_i} + \tfrac{1}{m}H_i) + (c_i - 1 - \tfrac{1}{m}\operatorname{mult}_{V_i} H_i)E_{i+1}.$$

Now note that $\operatorname{mult}_{V_i} H_i \le m$: indeed, $\operatorname{mult}_x H_i \le m$ for every $x \in X_i$ since this is what we have assumed for $i = 0$ and multiplicity does not increase after blowing up a smooth subvariety. Thus

$$c_i - 1 - \tfrac{1}{m}\operatorname{mult}_{V_i} H_i \geq 0, \qquad (5.11.2_i),$$

and so

$$K_{X_{i+1}} + \tfrac{1}{m}H_{i+1} \equiv f_{i+1}^*(K_{X_i} + \tfrac{1}{m}H_i) + (\text{effective divisor}). \qquad (5.11.3_i)$$

Now assume by induction that $(5.11.1_i)$ holds. (One can start with the base case $i = 0$ where $g_0 : X_0 \to X_0$ is the identity.) Substituting $(5.11.1_i)$ into $(5.11.3_i)$ we get that

$$\begin{aligned} K_{X_{i+1}} + \tfrac{1}{m}H_{i+1} &\equiv f_i^*\big(g_i^*(K_X + \tfrac{1}{m}H) + (\text{effective divisor})\big) \\ &\qquad + (\text{effective divisor}) \\ &= g_{i+1}^*(K_X + \tfrac{1}{m}H) + (\text{effective divisor}). \end{aligned}$$

This completes the proof. □

REMARK 5.13. It is not necessary to use resolution of singularities in the proof of Proposition 5.11. We are interested only in codimension one phenomena, so we can work always on the smooth locus of the varieties we consider. Indeed, to consider what happens at a divisor E over X, we look at the center Z of E on X; throwing away the nonsmooth locus Z, we then then blow it up inside the corresponding open set of X. If the center of E on the resulting blowup is not a divisor, we repeat the process. In order to make the proof of Proposition 5.11 go through, we need only know that eventually the center of E on some blowup is a divisor. We used this trick before in the proof of Abhyankar's lemma; see Remark 4.27 and especially §4.29.

Examining the proof, we see that Proposition 5.11 is in fact sharp for surfaces:

PROPOSITION 5.14. *Let H be a linear system on a smooth variety X. A codimension two subvariety W of X is a maximal center of $(X, \frac{1}{m}H)$ if and only if the multiplicity of H at each point of W is greater than m.*

PROOF. It suffices to show that if the multiplicity of H along a codimension two subvariety W is greater than m, then W is a maximal center of $(X, \frac{1}{m}H)$. First assume that W is smooth. If $f : Y \to X$ denotes the blow up of W, then

$$f^*H = f_*^{-1}H + (\operatorname{mult}_W H)\cdot E \qquad \text{and} \qquad K_Y = f^*K_X + E,$$

where E is the exceptional divisor. Thus E appears with negative coefficient in the expression

$$K_Y + \tfrac{1}{m} f_*^{-1} H - f^*(K_X + \tfrac{1}{m} H),$$

and so W is a maximal center.

The argument easily adapts to the case where W is not smooth. If we instead let f denote the blowup of the smooth part of W inside the corresponding open set of X, we have a map $f : Y \to X$ and an exceptional divisor E for which the above argument applies verbatim. $\square$

EXAMPLE 5.15. In dimensions greater than two, maximal centers are much more subtle. For instance, consider the linear system H on $\mathbb{A}^n$ spanned by the divisors $D_1, \dots, D_n$ where D_i is defined by the vanishing of the polynomial $x_i^{(n-1)m}$. The pair $(\mathbb{A}^n, \frac{1}{m}H)$ does not have any maximal centers (as can be checked after blowing up the origin; see also Exercise 6.9). However, the multiplicity at the origin is $(n-1)m$, which is greater than m as soon as n is three or more.

The next corollary gives a sufficient condition for the existence of maximal centers, showing that Example 5.15 is extremal.

COROLLARY 5.16. *Let H be a mobile linear system on a smooth variety X of dimension n. If H has multiplicity greater than $(n-1)m$ at some point P of X, then P is a maximal center of $(X, \frac{1}{m}H)$.*

PROOF. Let $f : X' \to X$ be the blow up of P. We have

$$K_{X'} + \tfrac{1}{m} H \equiv f^*(K_X + \tfrac{1}{m} H) + (n - 1 - \tfrac{1}{m} \operatorname{mult}_P H)E,$$

so the proof is complete. $\square$

EXERCISE 5.17. Fix positive integers a, b, and m, and fix a finite dimensional vector subspace of the space spanned by the monomials

$$\{x^i y^j z^k \mid ai + bj + k > m(a+b)\}.$$

Consider the linear system H whose members are the corresponding zero sets. Show that $(\mathbb{A}^3, \frac{1}{m}H)$ has a maximal center at the origin. (Hint: use the birational map $g : (u, v, w) \mapsto (uw^a, vw^b, w)$ of $\mathbb{A}^3$.)

REMARK 5.18. The simplest case of Exercise 5.17, when a and b are both one, recovers Corollary 5.16 in the dimension three case.

Considering the case where a is $m+1$ and b is one, Exercise 5.17 shows that the origin is a maximal center for the pair $(\mathbb{A}^3, \frac{1}{m}H)$, where H is the linear system on $\mathbb{A}^3$ spanned by the divisors obtained as the zero sets of the functions

$$x^{m+1}, \quad y^{(m+1)^2}, \quad \text{and} \quad z^{(m+1)^2}.$$

Here the multiplicity of H at P is $m+1$, the smallest allowed by Proposition 5.11. The small overall multiplicity is being compensated for by the high multiplicity in the variables y and z.

Exercise 5.17 gives a sufficient condition for the origin in $\mathbb{A}^3$ to be a maximal center of a linear system given by polynomials of a certain weighted degree (here $\operatorname{wt} x = a$, $\operatorname{wt} y = b$ and $\operatorname{wt} z = 1$). It is tempting to try to find similar sufficient conditions when we allow the weight of z to be arbitrary. Surprisingly, however, it turns out that the condition of Exercise 5.17 completely characterizes zero-dimensional maximal centers on a smooth threefold:

THEOREM 5.19. *Let P be a zero-dimensional maximal center of $(X, \frac{1}{m}H)$ where X is a smooth threefold. Then one can choose local coordinates (x, y, z) at P and natural numbers a and b such that the local equations for members of H can be written as*

$$\sum_{ai+bj+k>m(a+b)} a_{ijk} x^i y^j z^k.$$

The proof of Theorem 5.19 is unfortunately beyond our scope here. It follows from a recent result of Kawakita (2001), using a theory of Corti (1995). This result was first conjectured in Corti (2000).

Theorem 5.19 gives a complete local description of maximal centers on a smooth threefold, but in concrete questions it can still be quite hard to apply. One of the main practical difficulties is that the local coordinates (x, y, z) in Theorem 5.19 are not necessarily linear with respect to our global coordinates.

Fortunately, the next numerical result, already proved in Iskovskih and Manin (1971), is sufficient to study quartic threefolds.

THEOREM 5.20. *Let H be a mobile linear system on a smooth threefold X and let P be a zero-dimensional maximal center of $(X, \frac{1}{m}H)$. Let H_1 and H_2 be two members of H without a common irreducible component and let S be a smooth surface through P not containing any of the irreducible components of $H_1 \cap H_2$. Then the local intersection number $(H_1 \cdot H_2 \cdot S)_P$ is defined and greater than $4m^2$.*

Theorem 5.20 contains a deeper numerical result than the simple multiplicity bound provided by Proposition 5.11. Indeed, to prove it, we develop considerable machinery in Chapter 6. On the other hand, it is worth pointing out that if one is willing to assume Theorem 5.19, then Theorem 5.20 follows quite easily.

EXERCISE 5.21. Prove that Theorem 5.19 implies Theorem 5.20.

5.3 Birationally rigid Fano varieties

In this section, we use the Noether–Fano method to construct families of Fano varieties such that *no* smooth member is rational.

In general, it seems that the smaller the self-intersection of the canonical class (or the *degree*) of the Fano variety, the easier to apply the Noether–Fano method. We saw this in Corollary 2.12, where the Noether–Fano method was used to show that a degree one Del Pezzo surface of Picard number 1 is not rational. Every smooth degree one Fano surface is isomorphic to a degree six surface in the weighted projective space $\mathbb{P}(1, 1, 2, 3)$, as we saw in Theorem 3.36. This leads to the following higher dimensional analog of Corollary 2.12.

THEOREM 5.22. *For any integer c greater than one, let X be a smooth hypersurface of degree $4c + 2$ in the weighted projective space $\mathbb{P}(1^{2c}, 2, 2c + 1)$. (According to custom, the notation $\mathbb{P}(1^{2c}, 2, 2c + 1)$ means that we have $2c$ coordinates with weight 1.) Then X is a degree one Fano variety and, moreover,*

1. *X is not rational,*
2. *every birational self-map of X is an automorphism, and*
3. *any birational map from X to a smooth projective Fano variety of Picard number one is an isomorphism.*

The simplest example, when $c = 2$, produces a degree six hypersurface in $\mathbb{P}(1, 1, 1, 1, 2, 5)$. This example has dimension four. Other values of c give examples of every even dimension ≥ 4. As the following Exercise shows, it is not an accident that we miss the odd dimensions: there are no odd-dimensional Fano varieties of degree one.

EXERCISE 5.23. Let X be a smooth projective variety of odd dimension n. Prove that the self-intersection number K_X^n is even.

More generally, for any relatively prime natural numbers b and c greater than one, the general hypersurface of degree bc in the weighted projective space $\mathbb{P}(1^{(b-1)(c-1)}, b, c)$ is a smooth Fano variety of dimension $(b - 1)(c - 1)$ and degree one. It is very likely that all these hypersurfaces are nonrational and have all the rigidity properties listed in Theorem 5.22. Here we treat only the simplest series, where $b = 2$.

Theorem 5.22 follows easily from the most elementary numerical property of maximal centers, Proposition 5.11. However, we first need to have a good understanding of the geometry of our hypersurfaces. We now study this geometry in slightly greater generality.

5.24. HYPERSURFACES OF DEGREE $4c + 2$ IN $\mathbb{P}(1^n, 2, 2c + 1)$. We start with the weighted projective space $\mathbb{P}(1^n, 2, 2c + 1)$ where the notation indicates that we have n coordinates with weight 1. These weight one coordinates are denoted by $x_1, \ldots, x_n$, the weight two coordinate by y and the weight $2c + 1$ coordinate by z. By Exercise 3.51, the weighted projective space has two singular points $(0 : \cdots : 1 : 0)$ and $(0 : \cdots : 0 : 1)$.

Let X be a smooth hypersurface defined by a weighted homogeneous polynomial F of weighted degree $4c + 2$. We want to assume that X misses both singular points of the ambient projective space; this is equivalent to assuming that y^{2c+1} and z^2 both appear in F with nonzero coefficients. After a change of coordinates we can assume that these coefficients are 1 and by completing the square we can eliminate all other terms involving z. Thus we can write the defining equation F for X in the form

$$z^2 + y^{2c+1} + f_2(x_1, \ldots, x_n)y^{2c} + \cdots + f_{4c+2}(x_1, \ldots, x_n), \quad (5.24.1)$$

where the f_i are homogeneous of degree i.

In this case, we can carry out local computations on X as a hypersurface in smooth affine coordinates. Indeed, consider the coordinate charts U_i defined by the non-vanishing of x_i (see §3.50); the U_i all miss the singular points of $\mathbb{P}(1^n, 2, 2c + 1)$. Furthermore, there is only one point of X, namely

$$P_\infty = (0 : \cdots : 0 : -1 : 1),$$

that is not in the union of the U_i. Note that P_∞ lies in both U_y and U_z (the charts defined by the non-vanishing of the y and z coordinates respectively), and that the intersection $U_{yz} = U_y \cap U_z$ misses both singular points of $\mathbb{P}(1^n, 2, 2c + 1)$. Thus, in carrying out local computations about X, we can work always in the smooth charts U_i (for $i = 1, \ldots, n$) and U_{yz} and ignore the singular points of $\mathbb{P}(1^n, 2, 2c + 1)$ completely.

It is helpful to have explicit equations for X in the charts $U_1, \ldots, U_n$ and U_{yz}.

We first look at X in one of the U_i charts. As in §3.50, U_i is isomorphic to $\mathbb{A}^{n+1}$ with "standard" coordinate functions $x_j x_i^{-1}$ (for $j = 1, \ldots, n$ omitting

i), yx_i^{-2}, and zx_i^{-2c-1}. For instance, in the chart U_n, we have affine coordinates

$$\begin{aligned} X_j &= x_j x_n^{-1} \quad \text{for } j = 1, \dots, n-1, \\ Y &= yx_n^{-2}, \quad \text{and} \\ Z &= zx_n^{-2c-1}, \end{aligned}$$

and the hypersurface $X \cap U_n$ is given by the vanishing of the polynomial

$$Z^2 + Y^{2c+1} + f_2(X_1, \dots, X_{n-1}, 1)Y^{2c} + \cdots + f_{4c+2}(X_1, \dots, X_{n-1}, 1).$$

To look at the equation of X in U_{yz}, first note that

$$U_{yz} \cong \operatorname{Spec} k[x_1, \dots, x_n, y, y^{-1}, z, z^{-1}]_{(0)},$$

where the subscript "(0)" denotes the zeroth graded piece of the corresponding algebra; see §3.50. Because of our choice of weights, the monomial $y^c z^{-1}$ has weight -1, so that

$$\begin{aligned} &k[x_1, \dots, x_n, y, y^{-1}, z, z^{-1}]_{(0)} \\ &\quad = k[x_1 y^c z^{-1}, \dots, x_n y^c z^{-1}, z^2 y^{-2c-1}, z^{-2} y^{2c+1}]. \end{aligned}$$

This means that $U_{yz} \cong \mathbb{A}^n \times (\mathbb{A}^1 \setminus \{0\})$, and we can choose as affine coordinates

$$X'_j = x_j y^c z^{-1} \quad \text{for } j = 1, \dots, n, \qquad \text{and} \qquad W = z^2 y^{-2c-1}.$$

In this chart, the hypersurface $X \cap U_{yz}$ is given by the affine equation

$$1 + W^{-1} + f_2(X'_1, \dots, X'_n)W + \cdots + f_{4c+2}(X'_1, \dots, X'_n)W^{2c+1} = 0.$$

Let H denote the "hyperplane class" on $\mathbb{P}(1^n, 2, 2c+1)$, that is, the class of divisors of degree one, as discussed in §3.52. In particular, the members of H are defined by the vanishing of linear forms in the degree one coordinates x_i.

Having now such an explicit algebraic description of our hypersurfaces, it is easy to verify the following basic properties.

PROPOSITION 5.25. *Let X be a hypersurface of degree $4c+2$ in the weighted projective space $\mathbb{P}(1^n, 2, 2c+1)$, and assume that X misses both singular points of the ambient space. Then X is a variety of dimension n and:*

1. *the canonical class of X is $K_X = (2c-1-n)H|_X$, where H is the "hyperplane class" as defined in §3.52.*
2. *the self-intersection number $(H|_X)^n = 1$;*
3. *the hypersurface X is smooth at a point $(a_1 : \cdots : a_n : 0 : a_{n+2})$ if and only if*
   (a) *$a_{n+2} \neq 0$, or*

(b) $f_{4c}(a_1, \dots, a_n) \neq 0$, *or*
(c) $(a_1, \dots, a_n)$ *is a smooth point of the hypersurface in* $\mathbb{P}^{n-1}$ *defined by* $f_{4c+2} = 0$;

4. *every singular point of* X *is a double point.*

PROOF. By Exercise 3.53, the canonical class of $\mathbb{P}(1^n, 2, 2c+1)$ is $(-n-2c-3)H$. Thus (1) follows immediately from the adjunction formula.

For (2), consider the divisors $H_1, \dots, H_n$ defined by the vanishing of $x_1, \dots, x_n$ respectively. Each is in the class of H and their unique intersection point on X is P_∞. On the chart U_{yz}, the H_i are defined by the vanishing of the X_i', so their intersection is transverse. This proves (2).

For (3), we compute locally on X using a coordinate chart U_i containing $(a_1 : \dots : a_n : 0 : a_{n+2})$, say $i = n$. In this chart, the hypersurface X is given by the vanishing of

$$F = Z^2 + Y^{2c+1} + f_2(X_1, \dots, X_{n-1}, 1)Y^{2c} + \dots + f_{4c+2}(X_1, \dots, X_{n-1}, 1),$$

and P is $(\frac{a_1}{a_n}, \dots, \frac{a_{n-1}}{a_n}, 0, \frac{a_{n+2}}{a_n})$. So $\frac{\partial F}{\partial Z}$ is zero at P if and only if $a_{n+2} = 0$, while

$$\left.\frac{\partial F}{\partial Y}\right|_P = f_{4c}\left(\frac{a_1}{a_n}, \dots, \frac{a_{n-1}}{a_n}, 1\right)$$

and

$$\left.\frac{\partial F}{\partial X_j}\right|_P = \frac{\partial f_{4c+2}(X_1, \dots, X_{n-1}, 1)}{\partial X_j}\left(\frac{a_1}{a_n}, \dots, \frac{a_{n-1}}{a_n}\right)$$

for $j = 1, \dots, n-1$. Now P is a smooth point if and only if one of the above polynomials does not vanish at P. Since f_{4c} and f_{4c+2} are homogeneous, this corresponds to the conditions stated in (3) of the lemma.

Finally (4) is a consequence of the presence of the z^2 term in the equation for X. $\square$

In understanding the geometry of X, a useful feature is the projection

$$\begin{aligned} p : X &\dashrightarrow \mathbb{P}^{n-1} \\ (x_1 : \dots : x_n : y : z) &\mapsto (x_1 : \dots : x_n). \end{aligned}$$

This projection is defined everywhere except at P_∞.

Abusing notation slightly, we now let H denote the linear system on X obtained by pulling back the hyperplane system on $\mathbb{P}^{n-1}$ (note that this is really $H|_X$ if we use the previous notation in which H denotes the "hyperplane" system on $\mathbb{P}(1^n, 2, 2c+1)$). For a point $P \in X$ let H_P denote the linear subsystem of those members passing through P, or equivalently, the pullback of the linear system of hyperplanes through $p(P)$ on $\mathbb{P}^{n-1}$. We want to understand the singularities of the members of H_P.

It is helpful to consider the fibers of p. Over the point $\mathbf{a} = (a_1 : \cdots : a_n)$, the fiber of p is the curve $C_{\mathbf{a}}$ in the weighted projective plane $\mathbb{P}(1, 2, 2c+1)$ defined by the vanishing of

$$z^2 + y^{2c+1} + f_2(\mathbf{a})x^2y^{2c} + \cdots + f_{4c+2}(\mathbf{a})x^{4c+2}.$$

For any point P of X other than P_∞, let C_P denote the fiber of p passing through P. In other words, we write C_P for $C_{p(P)}$.

LEMMA 5.26. *The curve C_P is the complete intersection of any independent spanning set of members of the linear system H_P. In particular, the base locus of H_P is the curve C_P. Furthermore, the intersection number $C_P \cdot H$ is one.*

PROOF. Since every fiber contains a point of the form $(a_1 : \cdots : a_n : 0 : 0)$, we might as well assume that P has this form. Now let $H_1, \ldots, H_{n-1}$ be the pull-backs of any set of $n-1$ hyperplanes in $\mathbb{P}^{n-1}$ cutting out the point $p(P) = (a_1 : \cdots : a_n)$. Then clearly C_P is the intersection of the divisors $H_1, \ldots, H_{n-1}$, and hence it is the intersection of any set of independent spanning divisors. Thus from Lemma 5.25(2), the intersection number $C_P \cdot H$ is one. $\square$

We now consider the geometry of the fibers in more detail. Computing locally on charts, it is easily verified that fiber $C_{\mathbf{a}}$ of p over $\mathbf{a}$ is smooth if and only if the polynomial

$$y^{2c+1} + f_2(\mathbf{a})y^{2c} + \cdots + f_{4c+2}(\mathbf{a})$$

has no multiple roots. Again, any singular point of $C_{\mathbf{a}}$ is a double point because of the presence of the z^2 term. Thus the point P is always either a smooth point or a double point of the curve C_P. These two cases behave quite differently, and we wind up treating them separately in the ensuing arguments.

PROPOSITION 5.27. *Assume that X is smooth and let P be any point of X other than P_∞.*

1. *If P is a smooth point of C_P, then every member of H_P is smooth at P;*
2. *If P is a double point of C_P, then there is a unique member of H_P singular at P. This singular member, which we denote by T_P, is precisely the pull-back of the tangent plane to the hypersurface $X \cap \mathbb{P}^{n-1}$ at $p(P)$ under the map p. (Here $\mathbb{P}^{n-1}$ is identified with the subvariety of $\mathbb{P}(1^n, 2, 2c+1)$ where both y and z vanish.)*

PROOF. By changing coordinates via $y \mapsto y - \alpha x_i^2$ appropriately, we may assume that $P = (a_1 : \cdots : a_n : 0 : a_{n+2})$.

Recall that P is a singular point of C_P if and only if a_{n+2} and $f_{4c}(\mathbf{a})$ are both zero.

Now we derive the conditions for a member of H to be smooth at P. Consider a member D of H_P given by the vanishing of $\sum b_i x_i$. As a subvariety of the weighted projective space, D is defined by the vanishing of the two polynomials

$$b_1 x_1 + \cdots + b_n x_n \quad \text{and}$$
$$z^2 + y^{2c+1} + f_2(x_1, \ldots, x_n) y^{2c} + \cdots + f_{4c+2}(x_1, \ldots, x_n).$$

Computing locally in one of the charts U_i containing P, we see that $P = (a_1 : \cdots : a_n : 0 : a_{n+2})$ is a smooth point of D if and only if the matrix

$$\begin{pmatrix} 2a_{n+2} & f_{4c}(\mathbf{a}) & \frac{\partial f_{4c+2}}{\partial x_1}(\mathbf{a}) & \cdots & \frac{\partial f_{4c+2}}{\partial x_n}(\mathbf{a}) \\ 0 & 0 & b_1 & \cdots & b_n \end{pmatrix}$$

has full rank.

Now say that P is a smooth point of C. In this case, $a_{n+2} \neq 0$ or $f_{4c}(\mathbf{a}) \neq 0$, so the matrix above has full rank for any choice of b_i. Thus every member of H_P is smooth at P.

If C_P is not smooth at P, then $a_{n+2} = 0$ and $f_{4c}(\mathbf{a})$ vanishes. Since X is smooth at P, some $\frac{\partial f_{4c+2}}{\partial x_i}(\mathbf{a})$ does not vanish, by Proposition 5.25. In this case, the matrix above has full rank except when the vector $(b_1, \ldots, b_n)$ is a scalar multiple of the vector $(\frac{\partial f_{4c+2}}{\partial x_1}(\mathbf{a}), \ldots, \frac{\partial f_{4c+2}}{\partial x_n}(\mathbf{a}))$. Thus the only member of H_P singular at P is the one defined by

$$\frac{\partial f_{4c+2}}{\partial x_1}(\mathbf{a}) x_1 + \cdots + \frac{\partial f_{4c+2}}{\partial x_n}(\mathbf{a}) x_n = 0.$$

This is precisely the tangent plane described in (2). The proof is complete. $\square$

As in many applications of the Noether–Fano method, a key point is to understand the various linear systems on the blowup of X.

PROPOSITION 5.28. *Assume that X is smooth and let $\pi : X' \to X$ denote the blowup of a point P other than P_∞. Let E denote the exceptional divisor of π and let C'_P denote the birational transform of the curve C_P. Then*

1. *If P is a smooth point of C_P, then the curve C'_P is a transverse intersection of any set of independent spanning divisors of the linear system $\pi^* H_P - E$. In particular, the base locus of $\pi^* H_P - E$ is C'_P.*
2. *If P is a double point of C, then the curve C'_P is the complete intersection of any $(n-2)$ general members of $\pi^* H_P - E$ and the birational transform of the singular member T_P of H_P. Furthermore, the base locus of $\pi^* H_P - E$ consists of two components: the curve C'_P and a line in E.*

PROOF. If P is a smooth point of C_P, then C_P is a transverse intersection of a set of spanning divisors $H_1, \ldots, H_{n-1}$ in H_P. Their birational transforms are still transverse and intersect in the birational transform of C_P. This proves (1).

We have to be a little more careful if P is a singular point of C_P. In this case, C_P is the complete intersection of T_P (as defined in Proposition 5.27) and $(n-2)$ general members $H_1, \ldots, H_{n-2}$ of H_P. The H_i intersect transversally, and so do their birational transforms. The delicate point comes when we take T_P.

Fix local coordinates $u_1, \ldots, u_n$ such that H_i is defined by $u_i = 0$ for $i = 1, \ldots n-2$. If T_P has local equation $t(u_1, \ldots, u_n) = 0$, then in a typical chart its birational transform T'_P is given by the equation

$$u_n^{-\operatorname{mult}_P t(u_1,\ldots,u_n)} \cdot t(u'_1 u_n, \ldots, u'_{n-1} u_n, u_n) = 0.$$

In this chart, the curve C_P is given by equations

$$t(0, \ldots, 0, u_{n-1}, u_n) = u_1 = \cdots = u_{n-2} = 0.$$

Its birational transform C'_P is given by equations

$$u_n^{-\operatorname{mult}_P t(0,\ldots,0,u_{n-1},u_n)} \cdot t(0, \ldots, 0, u_{n-1}, u_n) = u'_1 = \cdots = u'_{n-2} = 0.$$

Therefore, if $\operatorname{mult}_P t(u_1, \ldots, u_n) = \operatorname{mult}_P t(0, \ldots, 0, u_{n-1}, u_n)$ then the restriction of the equation of T'_P gives the equation of C'_P but not otherwise. In the case where P is a double point of C_P, both C_P and T_P have double points at P. This shows that C'_P is the complete intersection of $H'_1, \ldots, H'_{n-2}$, and T'_P. Furthermore, in this case, the base locus of $\pi^* H_P - E$ is given by

$$u_n^{-1} \cdot t(u'_1 u_n, \ldots, u'_{n-1} u_n, u_n) = u'_1 = \cdots = u'_{n-2} = 0.$$

But since u_n divides $t(u'_1 u_n, \ldots, u'_{n-1} u_n, u_n)$ exactly twice, the base locus is a union of the line L (where $u'_1 = \cdots = u'_{n-2} = 0$) and T'_P. □

The key numerical consequences of this result are gathered below. It is convenient to use the notion of *nef* divisors. By definition, a $\mathbb{Q}$-Cartier divisor on a projective variety is nef if it has non-negative intersection number with every curve. (Nef replaces the old and misleading terminology "numerically effective". It should be thought of as the abbreviation of "numerically free".)

PROPOSITION 5.29. *Assume that X is smooth and let $\pi : X' \to X$ denote the blowup of any point P in X other than P_∞. Let E denote the exceptional divisor of π. Then*

1. *If P is a smooth point of C_P, then $\pi^* H_P - E$ is nef.*
2. *If P is a double point of C_P, then $\pi^* H_P - \frac{1}{2} E$ is nef.*

Proof. Let e denote the multiplicity of C at P, so that $e = 1$ in case (1) and $e = 2$ in case (2). Let L be any line contained in E. We first claim that for any P,

$$(\pi^* H_P - \tfrac{1}{e}E) \cdot C'_P = 0, \tag{5.29.3}$$

where C'_P denotes the birational transform of the curve C_P, and also

$$(\pi^* H_P - \tfrac{1}{e}E) \cdot L = \tfrac{1}{e}. \tag{5.29.4}$$

Indeed, formula (5.29.3) follows from the equalities $H \cdot C_P = 1$ (see Lemma 5.26) and $C'_P \cdot E = e$, while formula (5.29.4) holds because $\pi^* H \cdot L = 0$ and $E \cdot L = -1$.

Now to show that $\pi^* H_P - \frac{1}{e}E$ is nef, we must check that its intersection number with any effective curve on X is non-negative. This is immediate for any curve outside its base locus, so we focus on curves contained in the base locus.

In case (1), the base locus is exactly C'_P by Proposition 5.28, and the non-negativity follows from (5.29.3).

In case (2), we write $\pi^* H_P - \frac{1}{2}E$ as $(\pi^* H_P - E) + \frac{1}{2}E$. The base locus of $(\pi^* H_P - E)$ consists of two curves, C'_P and some line L contained in E. Thus for any curve D other than C'_P and not contained in E, we have $(\pi^* H_P - \frac{1}{2}E) \cdot D$ is non-negative, as needed. Also $(\pi^* H_P - \frac{1}{2}E) \cdot C'_P$ is non-negative by (5.29.3), since C'_P is not contained in E. It only remains to check non-negativity for curves contained in E. But because E is a projective space, any such curve is numerically equivalent to mL, for some positive integer m. It follows that

$$(\pi^* H_P - \tfrac{1}{2}E) \cdot mL = (\pi^* H_P - E) \cdot mL + \tfrac{1}{2}E \cdot mL = \tfrac{m}{2} - \tfrac{m}{2} = 0.$$

This completes the proof. □

Theorem 5.30. *Let X be a smooth hypersurface of degree $4c + 2$ in the weighted projective space $\mathbb{P}(1^n, 2, 2c+1)$, and let D be any reduced irreducible member of the linear system $mH|_X$. Then the multiplicity of D at any point P of X is at most m, except when P is a singular point of C_P and D is the unique singular member of the linear system H_P.*

Proof. The case $P = P_\infty$ is easy. Indeed, choose any fiber C_Q not contained in D. Then $(D \cdot C_Q) = m$ by Lemma 5.26. On the other hand, this intersection number is at least as big as the product $\operatorname{mult}_P C_Q \cdot \operatorname{mult}_P D$. Thus $\operatorname{mult}_P D \le m$.

Now assume that P is not P_∞. Again, there are two cases to consider, depending on whether or not P is a smooth point of C_P. However, the arguments are similar and can be combined by setting $e = \operatorname{mult}_P C_P$ as in the proof of

Proposition 5.29. Thus e is either one or two, and in the case $e = 2$, the divisor D is not the singular divisor T_P (see Proposition 5.27 for an explicit description of T_P). To further emphasize the similarity in the arguments, in the $e = 1$ case, we pick any divisor in H_P other than D and call it T_P.

Let $\pi : X' \to X$ denote the blowup of P, and let D' and T'_P denote the birational transforms of D and T_P respectively. By assumption $D' \cdot T'_P$ is an effective codimension two-cycle on X'. The base locus of the linear system $\pi^* H_P - E$ is one-dimensional by Proposition 5.28, thus if $H'_3, \ldots, H'_{n-1}$ are general members then

$$D' \cdot T'_P \cdot H'_3 \cdots H'_{n-1}$$

is an effective curve on X'. So by Proposition 5.29 the intersection number

$$D' \cdot T'_P \cdot H'_3 \cdots H'_{n-1} \cdot (\pi^* H_P - \tfrac{1}{e(P)} E) \tag{5.30.1}$$

is non-negative.

Now we compute the number (5.30.1) in a different way. Set $s = \operatorname{mult}_P D$. Since D' is linearly equivalent to $\pi^* mH - sE$, we get a rational equivalence

$$\begin{aligned} D' \cdot T'_P \cdot H'_3 \cdots H'_{n-1} &\sim m(\pi^* H - E) \cdot T'_P \cdot H'_3 \cdots H'_{n-1} \\ &+ (m - s)E \cdot T'_P \cdot H'_3 \cdots H'_{n-1}. \end{aligned}$$

By Proposition 5.28, we know that C'_P is the complete intersection of $T'_P, H'_3, \ldots H'_{n-2}$ and one more general member of $\pi^* H_P - E$, thus

$$D' \cdot T'_P \cdot H'_3 \cdots H'_{n-1} \sim mC'_P + (m - s)eL,$$

where L is any line in E. Using the equalities (5.29.3) and (5.29.4), we get that

$$\begin{aligned} D' \cdot T'_P \cdot H'_3 \cdots H'_{n-1} \cdot (\pi^* H_P - \tfrac{1}{e} E) &= m(C'_P \cdot (\pi^* H_P - \tfrac{1}{e} E)) \\ &+ (m - s)e(L \cdot (\pi^* H_P - \tfrac{1}{e} E)) = m - s. \end{aligned}$$

Comparing with (5.30.1) we get that $m \geq s$. □

5.31. The Picard group of a hypersurface in weighted projective space. Before getting down to the very short proof of Theorem 5.22, there is one more, unfortunately rather thorny, problem left. Namely, we need to know that our hypersurfaces have Picard number one. This is a special case of a quite general result.

THEOREM 5.32 (Lefschetz theorem on Picard groups). *The Picard group of any smooth hypersurface in a weighted projective space of dimension at least four is generated by the hyperplane class.*

PROOF. This is a well known result which we unfortunately have to accept on faith. There are several methods to prove this, none very elementary. Let us start with the classical case of hypersurfaces in ordinary projective spaces.

1. By the Lefschetz theorem on hyperplane sections (see Griffiths and Harris 1978, 1.2), the pull-back map $\mathbb{Z} \cong H^2(\mathbb{CP}^n, \mathbb{Z}) \to H^2(X(\mathbb{C}), \mathbb{Z})$ is an isomorphism when n is at least four. Furthermore, $H^1(X(\mathbb{C}), \mathbb{Z})$ is zero, so that, by Hodge theory, $H^1(X, \mathcal{O}_X)$ is zero as well. This implies that the Picard group of X is $H^2(X(\mathbb{C}), \mathbb{Z})$, and hence is isomorphic to $\mathbb{Z}$.
2. The affine cone $C_X \subset \mathbb{C}^{n+1}$ over X is an isolated hypersurface singularity. By Milnor (1968), its link, by which we mean its intersection with a small sphere around the origin, is $n-2$-connected. So when n is at least four, it is two-connected. From this it follows that the singularity of the vertex of the cone is factorial. This is equivalent to the Picard group being generated by the hyperplane bundle.

Both of the above arguments rely on topology and only work over $\mathbb{C}$. As in Grothendieck (1962), the first approach can be made to work in arbitrary characteristic by the method of formal schemes. The second approach can also be made to work in any characteristic since an isolated hypersurface singularity of dimension at least four is factorial in any characteristic. This was first proved in Grothendieck (1962); see Call and Lyubeznik (1994) for an easier treatment.

The topological proofs are usually written only for ordinary projective spaces, but the more algebraic approach of Grothendieck (1962) works in the general setting. □

REMARK 5.33. It is worth noting that the weighted projective case can almost be reduced to the classical case of ordinary projective space. Indeed, there is a finite, surjective morphism

$$q : \mathbb{P}^n \to \mathbb{P}^n(a_0, \dots, a_n) \quad \text{given by} \quad (x_0 : \cdots : x_n) \mapsto \left(x_0^{a_0} : \cdots : x_n^{a_n}\right).$$

If X is a hypersurface in $\mathbb{P}^n(a_0, \dots, a_n)$, then its preimage $q^{-1}(X)$ is a hypersurface in $\mathbb{P}^n$, so it is enough to prove that the Picard number of $q^{-1}(X)$ is one. A slight problem arises in that $q^{-1}(X)$ may be singular even when X is smooth. From the point of view of the algebraic method of Grothendieck (1962) this does not matter.

In our case, X is given by the vanishing of

$$z^2 + y^{2c+1} + f_2(x_1, \ldots, x_n)y^{2c} + \cdots + f_{4c+2}(x_1, \ldots, x_n) = 0$$

and so $q^{-1}(X)$ is given by

$$Z^{4c+2} + Y^{4c+2} + f_2(X_1, \ldots, X_n)Y^{4c} + \cdots + f_{4c+2}(X_1, \ldots, X_n).$$

There are a few cases when X is smooth but $q^{-1}(X)$ is not, for instance when X is given by $z^2 + y^{2c+1} + yx_1^{4c} + x_2^{4c+2} + \cdots + x_n^{4c+2}$. In general, however, $q^{-1}(X)$ is smooth. To see this, it is enough to find one such example (say $z^2 + y^{2c+1} + x_1^{4c+2} + \cdots x_n^{4c+2}$), since smoothness is an open condition. At least for these smooth X, where $q^{-1}(X)$ is also smooth, the classical Lefschetz theorem alone implies that X has Picard number one.

Proof of Theorem 5.22. Consider a smooth degree $4c + 2$ hypersurface X in $\mathbb{P}(1^{2c}, 2, 2c + 1)$. By Theorem 5.32, the variety X has Picard number one. Also, by Proposition 5.25, the canonical class of X is $K_X = -H|_X$ and $K_X^{2c} = 1$. Thus K_X is a Fano variety of degree one.

We now prove statements (1), (2), and (3). Note that X is not isomorphic to $\mathbb{P}^{2c}$, since $-K_{\mathbb{P}^{2c}}$ has degree $(2c + 1)^{2c}$. Thus it suffices to prove (3).

According to the Noether–Fano method (especially Corollary 5.10), we need to prove that if G is a mobile linear system on X contained in the complete linear system $|-mK_X|$, then $(X, \frac{1}{m}G)$ does not have a maximal center. Suppose to the contrary that W is a maximal center of G. By Proposition 5.11 $\operatorname{mult}_W G > m$, so, in particular, $\operatorname{mult}_P D > m$ for every member D of G and every point P in W. This contradicts Theorem 5.30, since there are plenty of members D of G not equal to T_P. This proves Theorem 5.22. □

5.4 Quartic threefolds

In this section, we apply the Noether–Fano method to deduce the following threefold analogs of the theorems of Segre and Manin on cubic surfaces. By a *quartic threefold* throughout, we mean a hypersurface in projective four-space defined by a homogeneous equation of degree four.

Theorem 5.34. 1. *No smooth quartic threefold is rational.*
2. *Every birational self-map of a smooth quartic threefold is an automorphism (and hence a linear change of coordinates).*
3. *Any birational map from a smooth quartic threefold to a smooth projective Fano variety of Picard number one is an isomorphism.*

Both cubic surfaces and quartic threefolds are hypersurfaces embedded by their anti-canonical linear system. Since the Picard group of a quartic threefold always has rank one (as follows from Proposition 5.32), the fact that no quartic threefold is rational directly generalizes Segre's theorem that no cubic surface of Picard number one is rational. Note that Theorem 5.34 is stated without any explicit mention of the field of definition, and indeed the result holds over any field.

The theorem was first stated by Fano (1915), but it was proved rigorously by Iskovskih and Manin only half a century later (1971). The original proof has been reworked from various angles, and simplified several times over; see for example Pukhlikov (2000) and Corti (2000). More recently, singular quartic threefolds have also been studied; see Pukhlikov (1988), Corti and Mella (2002), Mella (2003).

REMARK 5.35. Before embarking on the proof of Theorem 5.34, we point out that in contrast to the smooth case, there are interesting birational self-maps of a quartic threefold with a double point p. The simplest is the birational involution τ defined by setting $\tau(q)$ to be the unique third intersection point of the line through q and p with the quartic. A more interesting birational involution σ is defined as follows. Fix a line ℓ on the quartic threefold passing through the double point p (such a line exists by Exercise 1.49).

For any point q on the quartic, the plane through q and ℓ intersects the quartic in the line ℓ together with a plane cubic. Since the cubic contains the marked point p, it admits the standard "sign changing" involution of an elliptic curve. Because q lies on this elliptic curve, we can define $\sigma(q) = -q$.

In fact, it turns out that in many cases the two involutions τ and σ generate the group of birational self-maps of the quartic threefold up to linear change of coordinates; see Pukhlikov (1988), and Mella (2003).

We now prove Theorem 5.34, assuming Theorem 5.20, which is proved in the next chapter. According to the Noether–Fano method developed in Section 1, we need to understand maximal centers on a quartic threefold. The curve maximal centers are easily handled by the multiplicity bound of Proposition 5.11, while the deeper Theorem 5.20 takes care of zero-dimensional maximal centers.

PROOF OF THEOREM 5.34. A smooth quartic threefold is not isomorphic to $\mathbb{P}^3$; indeed, the intersection number K^3 is -4 for the quartic but -64 for projective three-space. Because a smooth quartic threefold has Picard number one (Proposition 5.32), it therefore suffices to prove only the third statement.

We start the proof as for the theorems of Segre and Manin on cubic surfaces. Let X be a smooth quartic threefold and let $\phi : X \dashrightarrow X'$ be a birational map to any smooth Fano threefold with Picard number one. By the Noether–Fano inequalities, if ϕ is not an isomorphism, then there is a linear system H on X contained in $|-mK_X|$, with the property that the pair $(X, \frac{1}{m}H)$ has a maximal center.

In our proof of Theorem 2.1, we next composed ϕ with some explicitly understood maps in order to lower the value of m. In this case, as in Corollary 2.12, we will directly show that the assumption of the existence of a maximal center leads to a contradiction. There are two cases to consider.

Case 1: The center is a curve W on X. Then by Proposition 5.11, every member D of H has multiplicity greater than m along W. However the existence of even a single such D contradicts Lemma 5.36 below.

Case 2: The center is a closed point P on X. Then, according to Theorem 5.20, the local intersection number $(H_1 \cdot H_2 \cdot S)_P$ is greater than $4m^2$, where H_1 and H_2 are general members of a mobile linear subsystem of $|-mK_X|$ and S is a general smooth surface through P. However, on a quartic threefold $-K_X$ is the hyperplane class, so taking S to be a general hyperplane section through P, we compute that $H_1 \cdot H_2 \cdot S = 4m^2$. This contradiction proves Theorem 5.34, assuming Theorem 5.20. □

LEMMA 5.36. *Let $X \subset \mathbb{P}^n$ be a smooth hypersurface and $n \geq 4$. Let D be a member of the complete linear system $|mH|$, where H is the hyperplane class on X. Then the multiplicity of D along any curve contained in X is at most m.*

PROOF. Let W denote a curve contained in the divisor D on X. The idea is to calculate the intersection number $W \cdot D$. On the one hand this is just $m \deg W$, but below we find another expression bounding $W \cdot D$ that involves the multiplicity of D along W.

To compute $W \cdot D$, we use the trick of Severi of moving W away from D along a cone over W; see Fulton (1998, Example 11.4.1). Let Σ be the cone over W whose vertex v is a general point in the ambient projective space $\mathbb{P}^n$. The intersection of Σ with the hypersurface X consists of the curve W together with some residual curve Z, that is, $\Sigma \cdot X = W + Z$. Since $\deg \Sigma = \deg W$, we have

$$\begin{aligned} W \cdot_X D &= (W + Z) \cdot_X D - Z \cdot_X D = \Sigma \cdot X \cdot D - Z \cdot_X D \\ &= m \deg W \deg X - Z \cdot_X D, \end{aligned}$$

where "$\cdot_X$" denotes intersection product on X and the unadorned "$\cdot$" denotes intersection in $\mathbb{P}^n$.

Let $Z' \subset X$ be any curve none of whose irreducible components is contained in D and W' any subvariety of D. Then the intersection number $Z' \cdot_X D$ is bounded from below by the multiplicity of D along W' times the number of intersection points of Z with W'. In particular, when $Z' = Z$ and $W' = W$ as above, we have

$$Z \cdot_X D \geq (\text{mult}_W D)(\#(Z \cap W)).$$

Usually such a bound is uninteresting, but because W and Z lie on the same low degree surface (namely Σ), we expect to get a good bound. Indeed, as we compute below, $\#(Z \cap W) = (\deg W)(\deg X - 1)$ for a sufficiently general choice of the vertex of the cone Σ. Thus

$$m \deg W = W \cdot_X D \leq m \deg W \deg X - (\text{mult}_W D)(\deg W)(\deg X - 1).$$

Rearranging, we see that $\text{mult}_W D$ is at most m, as desired.

It remains to verify that for sufficiently general choice of the vertex of the cone Σ over W, the curves Z and W intersect in precisely $(\deg W)(\deg X - 1)$ points.

Since $n \geq 4$, we can assume that v is not on any of the tangent planes of X at singular points of W, not on any tangent line of W at a smooth point of W and not on any secant line of W. This implies that projection from v is an isomorphism on W.

Let $P \in W$ be a smooth point and L the line connecting P and the vertex v. Then P is a smooth point of Σ and $P \in Z \cap W$ if and only if the scheme theoretic intersection $L \cap (Z \cup W)$ has length at least two at P. Since $L \cap (Z \cup W) = L \cap X$, these are exactly the points in $R \cap W$, where $R \subset X$ is the ramification divisor of the projection $\pi_v : X \to \mathbb{P}^{n-1}$.

Let $x_0, \ldots, x_n$ be coordinates on $\mathbb{P}^n$ so that $v = (v_0 : \cdots : v_n)$ and suppose that the hypersurface X is defined by the vanishing of the homogeneous polynomial F in the x_i. A point $P = (p_0 : \cdots : p_n) \in X$ is a ramification point of π_v if and only if v is contained in the tangent plane $T_P X$. That is, if

$$\sum_i \frac{\partial F}{\partial x_i}(P)(v_i - p_i) = 0.$$

Note that

$$\sum_i p_i \frac{\partial F}{\partial x_i}(P) = \deg F \cdot F(P) = 0,$$

thus we get the simpler equation

$$\sum_i v_i \frac{\partial F}{\partial x_i} = 0.$$

Because X is smooth, the linear system on X spanned by the divisors defined by the partial derivatives $\frac{\partial F}{\partial x_i}$ is base point free. Hence, for general v, the divisor R, which is a general member of this linear system, must intersect W everywhere transversally. Since the degrees of the partial derivatives $\frac{\partial F}{\partial x_i}$ are all equal to $(\deg X - 1)$, there are a total of $(\deg X - 1)(\deg W)$ intersection points. □

6
Singularities of pairs

In the last chapter, we introduced the Noether–Fano method for proving nonrationality of certain varieties. In this chapter, we develop further practical techniques for carrying out this program.

To successfully use the Noether–Fano method, one needs to verify whether or not linear systems admit maximal centers. Recall that a maximal center is, roughly speaking, a place where there is a significant discrepancy between the birational transform and the pullback of a linear system. In this chapter, we develop tools for computing these discrepancies explicitly. The main issues we consider are the behavior of discrepancies under changing birational models, under finite morphisms, and under restriction to hypersurfaces.

As our main application, we prove Theorem 5.20 on the numerical consequences at isolated maximal centers on a threefold, which was used in our proof of the nonrationality of smooth quartic threefolds. The classical proof of Theorem 5.20 consisted of rather intricate computations on a suitable resolution of the singularities of the linear system, but we here place the result in the more conceptual framework of singularities of pairs.

The first three sections develop the basic theory of discrepancies and singularities of pairs. Our motivation is that a linear system has a maximal center if and only if the corresponding pair fails to be *canonical.* The important issues of how discrepancies can be computed from a log resolution and how they behave under finite maps are treated in the exercises.

In Section 4, we prove the very useful *inversion of adjunction* theorem, which illuminates how discrepancies behave under restriction to hypersurfaces. Inversion of adjunction is often helpful in carrying out inductive proofs with pairs, and, in fact, we later use it to reduce the proof of Theorem 5.20 to a statement about surfaces. This statement about surfaces follows from the formula for the log canonical threshold of a curve on a surface we derive in Section 5. In Section 6, we pull these details together into a proof of Theorem 5.20.

This is probably the most technically demanding chapter of the book, mainly because of the number of new concepts introduced. The proof of Theorem 6.32, used in the proof of inversion of adjunction, has been relegated to the Appendix because it is somewhat more technically challenging; in particular, it makes use of a recent refinement of the Grauert–Riemenschneider vanishing theorem, Theorem 6.57, for which we have included only a reference.

For simplicity, we assume throughout this chapter that all varieties are defined over a field of characteristic zero. While the main consequence, that no quartic threefold is rational, remains true in prime characteristic (see Pukhlikov, 2000), our proof of Theorem 5.20 makes free use of resolution of singularities and especially certain vanishing theorems valid only in characteristic zero.

6.1 Discrepancies

Discrepancies can be thought of as a more refined way to keep track of the information provided by maximal centers.

Let H be a mobile linear system on a smooth variety X. Recall that a maximal center of the pair (X, cH) is defined as the center on X of some prime divisor exceptional over X appearing with negative coefficient in the f-exceptional divisor

$$(K_Y - f^*K_X) - c(f^*H - f_*^{-1}H),$$

where $f : Y \to X$ is a proper birational morphism. The notion of discrepancy keeps track of not only whether some divisor appears with negative coefficient, but also of the actual value of the coefficient for each divisor lying over X.

Historically, discrepancies were usually defined for $\mathbb{Q}$-divisors, while we are interested in discrepancies of a single linear system with $\mathbb{Q}$-coefficients. Following Alexeev (1994), we present the theory of discrepancies for $\mathbb{Q}$-linear combinations of linear systems, a natural framework that incorporates both approaches.

NOTATION 6.1. Let (X, Δ) be a *pair* consisting of a normal variety X and a $\mathbb{Q}$-linear combination $\Delta = \sum a_i D_i$ of linear systems D_i on X. We say that Δ is *effective* if all the a_i are non-negative.

In applications one almost always considers pairs (X, Δ) where Δ is effective. In many papers effectivity is considered to be part of the definition. For inductive purposes, for instance in Exercise 6.9, it is sometimes useful to consider pairs where Δ is not effective, but the key theorems all need the effectivity assumption.

There is one small ambiguity in this notation. If D is a linear system, and m is an integer then mD could mean D formally multiplied by m or it could mean the linear system whose members are sums of m members of D. We will always mean the first of these, unless the opposite is explicitly claimed (as for instance at the end of this paragraph) but in all applications in these notes the difference does not matter.

Two extreme cases form the most important examples. The traditional setting is where each D_i has only a fixed part so that Δ can be interpreted as a $\mathbb{Q}$-divisor. The other extreme is when there is only one mobile linear system so that Δ has the form cH as above. (For concreteness, the novice reader may prefer work only with the classical case where Δ is a $\mathbb{Q}$-divisor, although there is little difference. In fact, the divisor case is sufficient for most of our purposes; see Exercise 6.24.)

Fix a birational morphism $f : Y \to X$ and a pair (X, Δ) as defined above. Roughly speaking, the discrepancy of the pair (X, Δ) along an f-exceptional divisor E is the coefficient of E in the difference divisor comparing two linear systems on X and Y:

$$(K_Y + f_*^{-1}\Delta) - f^*(K_X + \Delta). \tag{6.1.1}$$

The notation $f_*^{-1}\Delta$ makes sense as the $\mathbb{Q}$-linear combination $\sum a_i f_*^{-1} D_i$, but considerable care is needed in interpreting the expression (6.1.1), as we now explain.

To make this precise, first assume for simplicity that X is smooth, so that all divisors on X are Cartier. As discussed in Paragraph 5.2, the expressions

$$K_Y - f^*K_X \qquad \text{and} \qquad f^*D_i - f_*^{-1}D_i \tag{6.1.2}$$

make sense as well-defined f-exceptional divisors on Y, not merely as linear equivalence classes of divisors. Thus if $\Delta = \sum a_i D_i$ is a $\mathbb{Q}$-linear combination of linear systems D_i, the expression (6.1.1) is interpreted as the f-exceptional $\mathbb{Q}$-divisor

$$(K_Y - f^*K_X) - \sum a_i (f^*D_i - f_*^{-1}D_i). \tag{6.1.3}$$

If X is not smooth, we can make sense of the expressions in (6.1.2) whenever the linear systems K_X and D_i consist of $\mathbb{Q}$-Cartier divisors. In this case there are natural numbers m_i such that every member of the linear system $m_i D_i$ is Cartier; here $m_i D_i$ denotes the sum $D_i + \cdots + D_i$, m_i-times. Assume that $m_0 K_X$ is also Cartier. In this case, setting $m := \prod m_i$, the linear systems mK_X and mD_i all can be pulled back and then the expression (6.1.3) is interpreted to mean the f-exceptional $\mathbb{Q}$-divisor

$$\tfrac{1}{m}\left[(mK_Y - f^*(mK_X)) - \sum a_i(f^*(mD_i) - f_*^{-1}(mD_i))\right].$$

REMARK 6.2. *Throughout this chapter, we assume for simplicity that the linear systems K_X and D_i are always $\mathbb{Q}$-Cartier.* Every linear system is $\mathbb{Q}$-Cartier, for instance, on a variety X that is locally isomorphic to the quotient of a smooth variety by a finite group.

More generally, the expression (6.1.1) is meaningful as an f-exceptional $\mathbb{Q}$-divisor whenever $K_X + \Delta$ is $\mathbb{Q}$-Cartier, meaning that for some (equivalently, any) member of the classes K_X and D_i, the corresponding $\mathbb{Q}$-divisor $K_X + \sum a_i D_i$ is $\mathbb{Q}$-Cartier. See, for example, Kollár (1992). Although everything we prove here is valid in this setting, we do not need the theory in this generality.

DEFINITION 6.3. Let E be a prime exceptional divisor over X. The *discrepancy* of the pair (X, Δ) along E, denoted by $a(E, X, \Delta)$, is defined by the expression

$$(K_Y + f_*^{-1}\Delta) - f^*(K_X + \Delta) \equiv \sum_i a(E_i, X, \Delta)E_i, \tag{6.3.1}$$

where $E_1, \ldots, E_t$ denote the irreducible components of the exceptional divisor of some birational morphism $f : Y \to X$ such that E is one of the E_i.

We frequently write this formula as

$$K_Y + f_*^{-1}\Delta \equiv f^*(K_X + \Delta) + \sum_i a(E_i, X, \Delta)E_i, \tag{6.3.2}$$

or as

$$K_Y + f_*^{-1}\Delta + F \equiv f^*(K_X + \Delta). \tag{6.3.3}$$

In the latter case, the discrepancies are minus the coefficients of the exceptional divisors in F.

CAUTION 6.4. The expressions (6.3.1–3) should be interpreted only as shorthands for the ideas discussed in preceding paragraphs. If Y is complete, however, they do make sense as numerical equivalences of $\mathbb{Q}$-divisors, as shown by the next exercise.

EXERCISE 6.5. Let X be a normal, projective variety and let $\Delta = \sum a_i D_i$ be a linear combination of linear systems. Assume that K_X and the D_i are all $\mathbb{Q}$-Cartier. Let $f : Y \to X$ be a projective birational morphism, Y normal. Then there is a unique linear combination of linear systems Δ_Y such that

1. $K_Y + \Delta_Y$ is $\mathbb{Q}$-Cartier,

2. $K_Y + \Delta_Y$ is numerically equivalent to $f^*(K_X + \Delta)$, and
3. $f_* \Delta_Y = \Delta$.

(We emphasize that the third condition is equality, not just linear equivalence.)

REMARK 6.6. The discrepancy of a pair (X, Δ) along E is insensitive to the particular variety on which E appears. Indeed, suppose that $g : Y' \to X$ is another birational morphism from a normal variety Y' and that the birational transform E' on Y' is also a divisor. Then because the map $f \circ g^{-1}$ is an isomorphism in a neighborhood of E, it follows immediately that

$$a(E, X, \Delta) = a(E', X, \Delta).$$

This is the reason that the map f and the variety Y are suppressed in the notation for the discrepancy.

The following exercise shows that base point free linear systems can be added to Δ without affecting the discrepancies.

EXERCISE 6.7. Consider a $\mathbb{Q}$-linear combination of linear systems Δ on a normal variety X as defined in Paragraph 6.1. Let Δ' be any effective $\mathbb{Q}$-linear combination of linear systems on X. For any prime divisor E over X, show that

1. $a(E, X, \Delta) \geq a(E, X, \Delta + \Delta')$, and
2. the inequality is strict if and only if the center of E on X is contained in the union of the base loci of the components of Δ'.

REMARK 6.8. The definition of discrepancy can be extended to include all prime divisors over X, not just the exceptional ones. Let $f : Y \to X$ be a birational morphism and let D be a prime divisor on Y that is not f-exceptional. It is reasonable to define the discrepancy of (X, Δ) along D to be minus the multiplicity of $f_*^{-1}\Delta$ along D. More precisely, if $\Delta = \sum a_i D_i$, then

$$a(D, X, \Delta) = -\sum a_i \operatorname{mult}_D(f_*^{-1} D_i).$$

This is the same as $-\sum a_i \operatorname{mult}_{f_* D} D_i$, so Remark 6.6 on the invariance of discrepancies under birational maps is still valid in this generality. Note that the multiplicity of a linear system along a divisor D is simply the coefficient of D as a component of the fixed part of the system; the multiplicity along D is zero if D is not a fixed component.

Using this, it is convenient to rewrite formula (6.3.1) as

$$K_Y + \Delta_Y \equiv f^*(K_X + \Delta), \tag{6.8.1}$$

where Δ_Y is defined to be

$$(\text{mobile part of } f_*^{-1}\Delta) \; - \sum_{D \text{ arbitrary}} a(D, X, \Delta)D,$$

with the sum ranging over all the prime divisors of Y. By "mobile part" of a $\mathbb{Q}$-linear combination of linear systems $\Delta = \sum a_i D_i$, we mean the $\mathbb{Q}$-linear combination $\sum_i a_i M_i$, where M_i is the mobile part of the linear system D_i. Note that the non-exceptional part of $-\sum_D a(D, X, \Delta)D$ is simply $\sum a_i F_i$ where F_i is the fixed part of the system D_i.

We frequently refer to these formulas by saying:

$$\text{write } K_Y + \Delta_Y \equiv f^*(K_X + \Delta),$$

in which case it is understood that Δ_Y is chosen such that $f_*\Delta_Y = \Delta$.

The following exercise presents a trick for computing discrepancies that is quite useful; it often allows us to replace the pair (X, Δ_X) by a much simpler pair (Y, Δ_Y). We will use this repeatedly in Section 3.

EXERCISE 6.9. Let $f : Y \to X$ be a proper birational morphism between normal varieties. Let Δ_Y and Δ_X be $\mathbb{Q}$-linear combinations of linear systems on Y and X respectively such that

$$K_Y + \Delta_Y \equiv f^*(K_X + \Delta_X) \quad \text{and} \quad f_*\Delta_Y = \Delta_X.$$

Show that for any divisor F over X,

$$a(F, Y, \Delta_Y) = a(F, X, \Delta_X).$$

The following exercise is also helpful in computing discrepancies; we have essentially already made use of the idea in the proof of Proposition 5.11.

EXERCISE 6.10. Let $\Delta = \sum a_i D_i$ be a $\mathbb{Q}$-linear combination of linear systems on a smooth variety. Consider the blowup $p : Y \to X$ of a smooth subvariety Z of codimension c, and let E denote the resulting exceptional divisor. Show that

$$a(E, X, \Delta) = c - 1 - \sum_i a_i \cdot \operatorname{mult}_Z D_i.$$

6.11. TOTAL DISCREPANCY OF A PAIR. Let Δ be a $\mathbb{Q}$-linear combination of linear systems on a normal variety X, as defined in §6.1.

DEFINITION 6.12. The *discrepancy* of the pair (X, Δ) is the infimum, as E ranges over all *exceptional* divisors over X, of the discrepancies $a(E, X, \Delta)$. We denote this discrepancy by $\operatorname{discrep}(X, \Delta)$.

EXERCISE 6.13. Let (X, Δ) be a pair as above.

1. Show that the discrepancy of (X, Δ) is $-\infty$ if it is less than -1.
2. Show that the discrepancy of (X, Δ) is bounded above by one.

EXERCISE 6.14. Show that the discrepancy of a smooth variety is one, that is, $\operatorname{discrep}(X, 0) = 1$.

The next exercise easily follows from Exercise 6.9. Despite its simplicity, this exercise provides a very useful tool for computing discrepancies, as we will see in Section 3.

EXERCISE 6.15. Let $f : Y \to X$ be a birational morphism between normal varieties and let Δ_X and Δ_Y be $\mathbb{Q}$-linear combinations of linear systems on X and Y respectively such that

$$K_Y + \Delta_Y \equiv f^*(K_X + \Delta_X) \qquad \text{and} \qquad f_*\Delta_Y = \Delta_X.$$

Show that

$$\operatorname{discrep}(X, \Delta_X) = \min\{\operatorname{discrep}(Y, \Delta_Y), a(E, X, \Delta_X)\},$$

where E runs through all exceptional divisors of f.

6.2 Canonical and log canonical pairs

We are now ready to define the important concepts of canonical and log canonical singularities. Canonical singularities are those whose discrepancies are non-negative; in particular, the pair (X, cH) has no maximal center if and only if it is canonical. It turns out to be useful to consider pairs with other discrepancy bounds as well.

DEFINITION 6.16. Let Δ be a $\mathbb{Q}$-linear combination of linear systems on a normal variety X of dimension at least two. We say that pair (X, Δ) is

$$\left.\begin{array}{c} \textit{terminal} \\ \textit{canonical} \\ \textit{purely log terminal (or plt)} \\ \textit{log canonical (or lc)} \end{array}\right\} \text{if } \operatorname{discrep}(X, \Delta) \left\{\begin{array}{l} > 0, \\ \geq 0, \\ > -1, \\ \geq -1. \end{array}\right.$$

We also say that (X, Δ) is *Kawamata log terminal* (or *klt*), if all its discrepancies are greater than -1 (as opposed to only the discrepancies of exceptional divisors).

If X is a smooth curve, there are no exceptional divisors, so the above definition does not quite makes sense. In order to achieve consistency, we say that on a smooth curve C the pair (C, Δ) is log canonical (respectively klt, canonical) if for every point $P \in C$ we have $\operatorname{mult}_P \Delta \leq 1$ (respectively < 1, ≤ 0). Here plt coincides with log canonical and nothing is terminal.

In some proofs one needs to check these cases by hand. This is always easy and we skip this step.

These notions tend to be useful only if Δ is an effective combination. However, for inductive purposes it is frequently convenient to allow negative coefficients in Δ.

HISTORICAL REMARK 6.17. Canonical singularities (with $\Delta = 0$) were first defined by Reid (1979) in the context of varieties of general type. Recall that X is of general type if there is a positive integer m such that the linear system $|mK_X|$ defines a map which is birational onto its image. If, in addition, the *canonical ring* $R(X, K_X) = \oplus_{n\geq 0} H^0(X, nK_X)$ is finitely generated (as it is conjectured to be always), then X has a *canonical model*

$$\bar{X} := \operatorname{Proj} R(X, K_X).$$

Canonical singularities are precisely those that appear on canonical models.

Terminal singularities appeared later as the *smallest* class of singularities where the minimal model program can be carried out. See Kollár and Mori (1998) for discussions of the minimal model program. After canonical singularities were defined, it took quite some time, and nontrivial thinking, to realize the importance of defining terminal singularities. See Reid (2000) for the history of these ideas.

The birational geometry of pairs (X, Δ), where Δ is a reduced divisor on X, began with the Iitaka program to study the birational geometry of open varieties. Let U be an open variety and let X be a compactification of U obtained by adding a divisor Δ. The *log plurigenera*, that is, the dimensions of $H^0(X, n(K_X + \Delta))$, are invariants of U, independent on the choice of the compactification X, provided that the pair (X, Δ) has log canonical singularities.

The utility of allowing fractional coefficients and eventually linear systems came into view later as singularities of pairs were noticed to arise naturally in many places, not merely inside the confines of the minimal model program, but also in classification theory, singularity theory, and algebraic geometry at large. Log canonical and log terminal singularities are subtle ideas at the intersection of several different lines of thought; this is the origin of the confusion in the subject, but also the reason why they are important.

Our motivation for studying canonical and log canonical singularities is to better understand maximal centers. Note first that Proposition 5.11 can be rephrased as follows: If the multiplicity of a mobile linear system H at a point

P is bounded above by m, then the pair $(X, \frac{1}{m}H)$ is canonical in a neighborhood of P. In general, we can define the multiplicity of a $\mathbb{Q}$-combination of linear systems $\Delta = \sum a_i D_i$ at a point P by

$$\operatorname{mult}_P \Delta = \sum a_i \operatorname{mult}_P D_i$$

where as usual, $\operatorname{mult}_P D_i$ is the multiplicity at P of a generic member of D_i. The statement and proof of Proposition 5.11 generalize to arbitrary pairs:

Exercise 6.18. Let Δ be an effective $\mathbb{Q}$-linear combination of linear systems on a normal variety X. If the multiplicity of Δ at a point P is at most one, then (X, Δ) is canonical in a neighborhood of P.

Our goal is to make use of canonical and log canonical pairs to also prove the harder numerical consequence of the existence of maximal centers, Theorem 5.20.

6.3 Computing discrepancies

It is cumbersome to verify whether or not a pair (X, Δ) is canonical directly from the definition because this involves understanding all birational morphisms $Y \to X$. Fortunately, it turns out that it is sufficient to understand a single morphism $Y \to X$, namely, a *log resolution* of the pair (X, Δ). The main idea is captured by Exercise 6.23, which allows us to check whether or not a pair is canonical (or log canonical, klt, etc.) by computing on a log resolution.

Definition 6.19. Let $\Delta = \sum a_i D_i$ be a $\mathbb{Q}$-linear combination of linear systems on a smooth variety. Then Δ is said to have *simple normal crossings* if, after decomposing each linear system D_i as $D_i = M_i + F_i$ where M_i is mobile and F_i is fixed, we have

1. each mobile part M_i is base point free, and
2. the sum of the fixed parts $\sum F_i$ forms a divisor whose support is in simple normal crossings, meaning each component is smooth and their intersections are all transverse.

Remark 6.20. If Δ is a $\mathbb{Q}$-divisor, this definition agrees with the usual notion of a simple normal crossings divisor.

Definition 6.21. A *log resolution* of the pair (X, Δ) is a proper birational morphism $f : Y \to X$ with Y smooth, such that $\operatorname{Ex} f$ has pure codimension one and $f_*^{-1}\Delta + \operatorname{Ex} f$ has simple normal crossings, where $\operatorname{Ex} f$ denotes the full exceptional set of f.

Every pair (X, Δ) admits a log resolution (at least in characteristic zero), by Hironaka's theorem on resolution of singularities. So in light of Exercise 6.15, the computation of the discrepancy of a pair (X, Δ_X) reduces to the computation of the discrepancy of a pair (Y, Δ_Y) where Δ_Y is in normal crossings. Furthermore, because base point free linear systems do not contribute to the discrepancy of the pair (Y, Δ_Y), we effectively reduce the computation of the discrepancy of a pair (X, Δ) to the case where X is smooth and Δ is a simple normal crossings $\mathbb{Q}$-divisor. Thus it is important to understand singularities of pairs in the normal crossings case.

EXERCISE 6.22. Let $\Delta = \sum a_i D_i$ be a simple normal crossings divisor on a smooth variety X, with all D_i distinct. Show that:

1. the pair (X, Δ) is log canonical if and only if $a_i \leq 1$ for every i;
2. the pair (X, Δ) is canonical if and only if $a_i \leq 1$ for every i and $a_i + a_j \leq 1$ whenever two distinct divisors D_i and D_j intersect;
3. the pair is klt if and only if $a_i < 1$ for every i;
4. the pair is plt if and only if $a_i \leq 1$ for every i and if two divisors intersect then at least one of them has coefficient less than one.

Now we can put all this together into the following characterization of singularities of pairs in terms of a log resolution. This provides a practical way to check for canonical singularities of which we make great use later.

EXERCISE 6.23. Let $f : Y \to X$ be a log resolution of (X, Δ). As in Remark 6.8, write

$$K_Y + (\text{mobile parts of } f_*^{-1}\Delta) + \Delta'_Y \equiv f^*(K_X + \Delta),$$

where

$$\Delta'_Y = \sum_D a(D, X, \Delta)D = \sum_i e_i D_i.$$

Here the sum is taken over all prime divisors of Y (although only finitely many have nonzero coefficients).

1. The pair (X, Δ) is log canonical if and only if $e_i \leq 1$ for every i.
2. The pair (X, Δ) is canonical if and only if $e_i \leq 1$ for every i, $e_i \leq 0$ for every exceptional divisor and $e_i + e_j \leq 1$ whenever two distinct divisors E_i and E_j intersect.
3. The pair (X, Δ) is klt if and only if $e_i < 1$ for every i.

4. Assume that the log resolution f is chosen so that the irreducible components of the fixed part of $f_*^{-1}\Delta$ are disjoint from each other (this is always possible by further blowing up). Then the pair (X, Δ) is plt if and only if $e_i \leq 1$ for every i with strict inequality for any e_i that is a coefficient of an exceptional divisor.

It is natural to wonder how the discrepancies of a linear system compare to the discrepancies of a general member. Note that if H is a base point free linear system on a smooth variety X, then the pair (X, cH) is always canonical, for any rational number c. However, for any divisor D (including a general member of H), the pair (X, cD) is *never* canonical if c is greater than one. Thus we can not expect a very naive comparison statement. By working on a log resolution, we can now clarify this issue:

Exercise 6.24. Let H be a mobile linear system on a smooth variety X.

1. If $c \leq 1$, then (X, cH) is canonical if and only if (X, cD) is canonical, where D is a general member of H.
2. (X, cH) is canonical if and only if $(X, \frac{c}{m}(D_1 + \cdots + D_m))$ is canonical, where $D_1, \ldots, D_m$ are general members of H, provided that $c \leq \frac{m}{2}$.

6.25. Discrepancies under finite morphisms. Finally, we also need to understand the behavior of discrepancies under pull-back by a finite morphism. The following result is essentially proven in Reid (1979).

Exercise 6.26. Let $g : X' \to X$ be a finite morphism between normal varieties, and assume that g is unramified outside a set of codimension two. Then

1. $(\deg g)(\operatorname{discrep}(X, \Delta) + 1) \geq (\operatorname{discrep}(X', g^*\Delta) + 1)$.
2. $\operatorname{discrep}(X', g^*\Delta) \geq \operatorname{discrep}(X, \Delta)$.
3. (X, Δ) is log canonical (respectively, klt) if and only if $(X', g^*\Delta)$ is log canonical (respectively, klt).

Exercise 6.27 (Cyclic quotients). Let ϵ be a primitive nth root of unity and let $\mathbb{Z}_n$ act on $\mathbb{C}^2$ by $(x, y) \mapsto (\epsilon^a x, \epsilon^b y)$. We always assume that a and b have no common factors: otherwise in effect we have an action of the smaller group $\mathbb{Z}_m$ for $m = n/\gcd(a, b)$. Let $R = R(n, a, b) \subset \mathbb{C}[x, y]$ denote the ring of invariants. Prove the following.

1. If $b = 0$, then $R = \mathbb{C}[x^n, y]$.
2. Let $d = \gcd(n, b)$ and write $n = md$. Then $\mathbb{Z}_d \subset \mathbb{Z}_n$ acts by $(x, y) \mapsto (\epsilon^{am} x, y)$. So $R(d, ma, 0) = \mathbb{C}[x^d, y]$ and $R(n, a, b) \cong R(m, ad, b)$. Thus we may always assume that $(a, n) = (b, n) = 1$. By a change of the generator of $\mathbb{Z}_n$ we can even assume that $a = 1$.

3. Prove that $\mathbb{C}[x, y]$ is a finite extension of R of degree n.
4. Set $X = \operatorname{Spec} R$ and let $p : \mathbb{C}^2 \to X$ be the induced morphism. Prove that X is smooth except at $p(0, 0)$ and
$$p : \mathbb{C}^2 \setminus \{(0, 0)\} \to X \setminus \{p(0, 0)\}$$
is unramified.
5. Conclude that $K_{\mathbb{C}^2} = p^* K_X$ and so cyclic quotients are klt by Proposition 6.26.
6. Find generators for $R(n, 1, n - 1)$, $R(n, 1, 1)$, and $R(2n + 1, 1, n)$. Find the relations among the generators.

6.4 Inversion of adjunction

We would like to understand the relationship between the singularities of a pair (X, Δ) in a neighborhood of a point P and the singularities of a pair $(S, \Delta|_S)$ where S is a hypersurface through P. It turns out that the natural comparison is between the discrepancies of the pairs $(S, \Delta|_S)$ and $(X, S + \Delta)$.

It follows rather immediately from the adjunction formula that the discrepancy of the pair (S, Δ_S) is bounded below by the discrepancy of $(X, \Delta + S)$. A deeper fact is that often the discrepancy of $(S, \Delta|_S)$ imposes a bound on the discrepancy of $(X, S + \Delta)$. This phenomenon is called *inversion of adjunction*. Inversion of adjunction is often useful in carrying out inductive arguments. Indeed, our motivation here is to use it to reduce the proof of Theorem 5.20 to a statement about a linear system of curves on a surface.

We begin with the easier adjunction statement from Kollár (1992, 17.2).

PROPOSITION 6.28. *Let Δ be a $\mathbb{Q}$-linear combination of linear systems on a normal variety X, and let S be a normal effective Cartier divisor not contained in the union of the base loci of the components of Δ. Then*
$$\operatorname{discrep}(S, \Delta|_S) \geq \operatorname{discrep}(X, S + \Delta).$$

PROOF. Fix a log resolution $f : Y \to X$ of the pair $(X, S + \Delta)$. Let S' denote the birational transform of S on Y, and let Δ' denote the birational transform of Δ on Y, with Δ'_f and Δ'_m denoting the fixed and the mobile parts of Δ' respectively. By further blowing up, if necessary, we may assume that the support of the $\mathbb{Q}$-divisor $S' + \Delta'_f$ is smooth; in particular, the intersection of S' and Δ'_f may be assumed to be empty. Note that the restriction f_S of f to S' is a log resolution of the pair $(S, \Delta|_S)$. In particular, Δ'_m restricts to the mobile part of $(f_S^{-1})_*(\Delta|_S)$ and these mobile parts consist of base point free linear systems on Y and S' respectively.

Now write

$$K_Y + (S' + \Delta'_m + \Delta'_f) \equiv f^*(K_X + S + \Delta) + \sum e_i E_i, \quad (6.28.1)$$

where the E_i are exceptional divisors for the map f. By the usual adjunction formula,

$$K_{S'} = (K_Y + S')|_{S'}, \quad \text{and} \quad (K_X + S + \Delta)|_S = K_S + \Delta|_S.$$

Noting that $\Delta'_f|_{S'} = 0$, we thus have that

$$\begin{aligned} K_{S'} + \big(\text{mobile part of } \big(f_S^{-1}\big)_*(\Delta|_S)\big) &\equiv \\ \equiv f_S^*(K_S + \Delta|_S) + \sum e_i(E_i \cap S'), \end{aligned} \quad (6.28.2)$$

where the E_i are all exceptional for f.

In this formula, note that we do not actually know how the birational transform of the divisor $\Delta|_S$ on S' looks, nor are we guaranteed that the divisors $E_i \cap S'$ are exceptional for f_S. But, because Δ'_m consists only of base point free linear systems, this formula *does* show that every discrepancy for $(S, \Delta|_S)$ which occurs among the exceptional divisors for f_S must also be a discrepancy for $(X, S + \Delta)$. Since f_S is also a log resolution of the pair $(S, \Delta|_S)$, it follows that the discrepancy of $(S, \Delta|_S)$ is no larger than that of $(X, S + \Delta)$. The proof is complete. □

There are two reasons why the inequality of Proposition 6.28 should not be an equality. First, it may happen that some divisor E_i is f-exceptional but that $E_i \cap S'$ is not f_S-exceptional. In this case, the divisor E_i contributes to the discrepancy of $(X, S + \Delta)$ but not to the discrepancy of $(S, \Delta|_S)$. This is a not a very serious issue, and it can be easily corrected by a slight change in the definition of discrepancy; see Kollár and Mori (1998, 5.46).

The more substantial reason why the inequality of Proposition 6.28 should not be an equality arises from exceptional divisors of $f : Y \to X$ which do not intersect S'. When we restrict to S', their contribution disappears. There is no obvious connection between the discrepancies of such divisors and the discrepancies of exceptional divisors of $S' \to S$.

Despite this, it has been conjectured that in many situations, the discrepancies of f and $f|_{S'}$ are more closely related than one might think, that is, there may be *inversion of adjunction.* The following theorem, proved by Shokurov in dimension three (Shokurov, 1992) and in Kollár (1992, 17.6–7) in general, is the first significant result along these lines.[1]

[1] Added in proof: Ein *et al* (2003) proves the general inversion of adjunction for X smooth.

THEOREM 6.29 (Inversion of adjunction). *Let Δ be an effective $\mathbb{Q}$-linear combination of linear systems on a smooth variety X and let S be any smooth hypersurface on X not contained in the union of the base loci of the components of Δ. Then*

1. *$(X, S + \Delta)$ is log canonical in a neighborhood of S if and only if $(S, \Delta|_S)$ is log canonical.*
2. *$(X, S + \Delta)$ is plt in a neighborhood of S if and only if $(S, \Delta|_S)$ is klt.*

REMARK 6.30. In fact, it is not necessary to assume that X and S are smooth. As long as X and S are normal, and S is Cartier, then the same argument goes through provided that $K_X + S + \Delta$ is $\mathbb{Q}$-Cartier.

EXERCISE 6.31. Prove the above Theorem when X is a surface.

In proving Theorem 6.29, the difficulty is to see why an exceptional divisor E over X with discrepancy less than -1 contributes to the discrepancy of $(S, \Delta|_S)$. The answer is that there is always a chain of exceptional divisors E_i, all with discrepancy less than -1, connecting E and the birational transform of S. This is the content of the next result, called the connectedness theorem, due to Shokurov in dimension three and Kollár in general.

THEOREM 6.32 (Kollár, 1992, 17.4). *Let D_X be an effective $\mathbb{Q}$-linear combination of linear systems on a normal variety X, and assume that $K_X + D_X$ is $\mathbb{Q}$-Cartier. Let $g : Y \to X$ be a log resolution of D_X and write*

$$K_Y + D + M \equiv g^*(K_X + D_X)$$

where M is the mobile part of $g_*^{-1} D_X$ and $D = -\sum d_i D_i$ the fixed part as in Remark 6.8. Set

$$F = \sum_{d_i \geq 1} d_i D_i .$$

Then every fiber of $g : \operatorname{Supp} F \to X$ is connected.

The proof of the connectedness theorem is postponed until the Appendix at the end of this chapter, because it uses some machinery beyond what we have been assuming in this text. Instead, we now assume the connectedness theorem and use it prove inversion of adjunction.

PROOF OF THEOREM 6.29. One direction follows easily from Proposition 6.28. Indeed, if $(X, S + \Delta)$ is log canonical, then Proposition 6.28 immediately implies that $(S, \Delta|_S)$ is log canonical. Similarly, if $(X, S + \Delta)$ is plt, then we know that discrep$(S, \Delta|_S)$ is greater than -1 as needed, and we need only verify that the discrepancies of the non-exceptional divisors are also greater than -1. From Exercise 6.23, because $(X, S+\Delta)$ is plt, the exceptional coefficients e_i are greater than -1 in Equation (6.28.1). Since the restriction of Δ'_f to S' is empty, we see that all non-exceptional contributions disappear and Equation (6.28.2) implies that $(S, \Delta|_S)$ is klt.

Thus we assume that (S, Δ_S) is log canonical (respectively, klt), and try to show $(X, S+\Delta)$ is log canonical (respectively, plt). The statement for log canonical singularities reduces to the klt case by Exercise 6.33, so we focus on proving (2).

Assume that $(S, \Delta|_S)$ is klt. Let $f : Y \to X$ be a log resolution of the pair $(X, S+\Delta)$, and adopt the same notation as in the proof of Proposition 6.28.

Write

$$K_Y + S' + \Delta'_f + \Delta'_m \equiv f^*(K_X + S + \Delta) + \sum e_i E_i \qquad (6.29.1)$$

where the E_i are all exceptional for f and $S' + \Delta'_f + \Delta'_m$ is the birational transform of $S+\Delta$, with Δ'_f and Δ'_m denoting the fixed and the mobile parts of the birational transform of Δ respectively. Because f is a log resolution, we know that Δ'_m consists of free linear systems and Δ'_f is a normal crossings $\mathbb{Q}$-divisor. As in the proof of Proposition 6.28, we also assume that Δ'_f is disjoint from S'.

According to Exercise 6.23, to show that $(X, S+\Delta)$ is plt in a neighborhood of S, we must show that e_i is greater than -1 for each E_i whose center on X meets S and that the coefficients of Δ'_f are at most one. By replacing X with a smaller neighborhood of S if necessary, we may assume that the centers on X of the divisors E_i and all the components of Δ'_f meet S.

To use the connectedness theorem 6.32, rewrite formula (6.29.1) in the form

$$K_Y + D + M \equiv f^*(K_Y + S + \Delta)$$

where

$$D = S' + \Delta'_f - \sum e_i E_i \quad \text{and} \quad M = \Delta'_m.$$

Let F be the part of D as in the statement of Theorem 6.32. Thus

$$F = S' + (\Delta'_f)_{\geq 1} - \sum_{e_i \leq -1} e_i E_i,$$

where the notation $(\Delta'_f)_{\geq 1}$ denotes the divisor consisting of only those components of Δ'_f appearing with coefficient at least one. We must show that $F = S'$.

As we have observed in the proof of Proposition 6.28, any exceptional E_i appearing in F must be disjoint from S', since $(S, \Delta|_S)$ is klt. Also, since we may assume that S' is disjoint from Δ'_f, clearly S' is disjoint from $(\Delta'_f)_{\geq 1}$ as well. But according to the connectedness theorem, the fibers of $f : \operatorname{Supp} F \to X$

are connected. This is a contradiction unless that $F = S'$, so the proof that $(X, S + \Delta)$ is plt is complete. □

EXERCISE 6.33. With notation as in Theorem 6.29, prove that $(X, S + \Delta)$ is log canonical if and only if $(X, S + c\Delta)$ is purely log terminal for every positive c less than one, and also that $(S, \Delta|_S)$ is log canonical if and only if $(S, c\Delta|_S)$ is klt for every positive c less than one. Use this to complete the proof of Theorem 6.29. (Hint: use Exercise 6.7.)

6.5 The log canonical threshold of a plane curve singularity

Let C be a curve on a smooth surface S. As we have seen in Exercise 6.18, the pair $(S, \frac{1}{m}C)$ is canonical if and and only if the multiplicity of C is at most m at every point of S. In this section, we derive a similar characterization of log canonical singularities in the surface case.

We formulate our criterion using the notion of the *log canonical threshold* of a curve C on a surface S at a point P. Since the pair (S, cC) is log canonical when c is zero but not when c is greater than one, there must be some "threshold point":

DEFINITION 6.34. The *log canonical threshold* of an effective divisor D on a smooth variety X is the rational number

$$\sup\{c : (X, cD) \text{ is log canonical}\}.$$

Clearly the log canonical threshold exists and is a positive number less than or equal to one. Computing the discrepancies on a log resolution, we see that the log canonical threshold is always a rational number. (Of course, one can also define the log canonical threshold of an effective $\mathbb{Q}$-linear combination of linear systems, but we do not need this here.)

We begin with a simple bound on the log canonical threshold.

LEMMA 6.35. *Let P be a point of multiplicity d on a curve C lying on a smooth surface S. Then the log canonical threshold of the pair (S, C) at P is at most $\frac{2}{d}$.*

PROOF. Let $p : S' \to S$ be the blowup of P and let E denote the exceptional curve. Setting C' to be the birational transform of C, we easily compute that

$$K_{S'} = p^*K_S + E \quad \text{and} \quad C' = p^*C - dE,$$

where d is the multiplicity of C at P.

For any rational number c, we can therefore write

$$K_{S'} + cC' = p^*(K_S + cC) + (1 - cd)E.$$

So if (S, cC) is log canonical, then $(1 - c \cdot d) \geq -1$, which means that

$$c \leq \tfrac{2}{d}.$$

This completes the proof. □

REMARK 6.36. In fact, the log canonical threshold is usually equal to $\frac{2}{d}$ in the situation of Lemma 6.35. To be precise, suppose that in local coordinates x and y near P, the curve C is given by some power series whose leading term is $f_d(x, y)$. Then the log canonical threshold of the pair (S, C) at P is equal to $\frac{2}{d}$ if and only if each root of f_d has multiplicity at most $\frac{d}{2}$. In more geometric language, this means that the log canonical threshold of (S, C) is $\frac{2}{d}$ if and only if each tangent to the curve C occurs with multiplicity at most $\frac{2}{d}$.

To see this, first note that

$$K_{S'} + E + \tfrac{2}{d}C' \equiv p^*(K_S + \tfrac{2}{d}C),$$

thus $(S, \frac{2}{d}C)$ is log canonical if and only if $(S', E + \frac{2}{d}C')$ is log canonical by Exercise 6.15. By inversion of adjunction, $(S', E + \frac{2}{d}C')$ is log canonical if and only if $(E, \frac{2}{d}(C'|_E))$ is log canonical. (It is sufficient to use the easy Exercise 6.31.) Hence we are reduced to a one dimensional question.

To treat this one dimensional case, we compute locally in coordinates on E. Let x and y be local coordinates for S near P and let x' and y' be coordinates for one of the standard affine charts of the blowup S'. In particular, $x = x'y'$ and $y = y'$. In this chart, the exceptional divisor E is defined by the vanishing of y', and thinking of x' as a local coordinate for E in this chart, the divisor $C'|_E$ is given by the vanishing of $f_d(x', 1)$. In other words, thinking of E as the projective line $\mathbb{P}^1$ with homogeneous coordinates x and y, the divisor $C'|_E$ is given by the zeros of the homogeneous polynomial $f_d(x, y)$. By Definition 6.16, log canonical on a smooth curve means that all multiplicties are at most one. This is equivalent to each root of $f_d(x, y)$ appearing with multiplicity at most $\frac{d}{2}$.

EXERCISE 6.37. Determine the log canonical threshold of the plane curve defined by the vanishing of $x^2 + y^{2d+1}$, by explicitly computing a log resolution. This curve vanishes to order two at the origin but its tangent cone is a double line. So by the previous remark, we expect the log canonical threshold to be strictly less than $\frac{2}{2} = 1$.

In theory, we know how to compute a log resolution for any plane curve. In practice, however, the combinatorial complexity of the resolution can be daunting, even for curves as simple as $x^b + y^a = 0$. On the other hand, there is another method for computing the log canonical threshold for such curves, which exploits the fact that the polynomial $x^b + y^a$ is homogeneous if we assign the weight a to x and the weight b to y. This method uses the notion of *weighted blowups*, a special case of toric birational morphisms. See Fulton (1993) for the general theory of toric geometry.

6.38. WEIGHTED BLOWUPS. Let $x_1, \dots, x_n$ be coordinates on $\mathbb{A}^n$. The usual blowup of the origin can be defined as the closure of the graph of the map

$$\mathbb{A}^n \dashrightarrow \mathbb{P}^{n-1} \quad \text{given by} \quad (x_1, \dots, x_n) \mapsto (x_1 : \cdots : x_n).$$

The blowing up morphism is then the natural projection onto the first factor $\mathbb{A}^n$.

Similarly, let $a_1, \dots, a_n$ be a sequence of positive integers. We always assume that they are relatively prime. We have a natural map

$$\mathbb{A}^n \dashrightarrow \mathbb{P}^{n-1}(a_1, \dots, a_n)$$

where $\mathbb{P}^{n-1}(a_1, \dots, a_n)$ is the weighted projective space as defined in §3.48, given by

$$(x_1, \dots, x_n) \mapsto \left(x_1^{a_1} : \cdots : x_n^{a_n}\right).$$

By definition, the weighted blow up of $\mathbb{A}^n$ with local coordinates $(x_1, \dots, x_n)$ and weights $a_1, \dots, a_n$ is the closure of the graph of this map. Again, the weighted blowing up morphism is given by projection onto the first factor $\mathbb{A}^n$. There is no loss of generality in assuming that the a_i are relatively prime. It should be emphasized, however, that the weighted blow up *does* depend on the local coordinates chosen.

Weighted blow-ups can be described in terms of charts in a manner similar to the standard blowup, but subject to a group action. Recall that if we fix one of the standard affine charts of the blowup of the origin in $\mathbb{A}^n$, the blowing up morphism is given there by

$$\pi_i : \mathbb{A}^n \to \mathbb{A}^n$$

$$(x'_1, \dots, x'_n) \mapsto (x'_i x'_1, \dots, x'_i x'_{i-1}, x'_i, x'_i x'_{i+1}, \dots, x'_i x'_n).$$

In other words, π_i is given by the formulas

$$x_j = x'_j x'_i \quad \text{if} \quad j \neq i \quad \text{and} \quad x_i = x'_i.$$

To describe the analog for weighted blow-ups, fix relatively prime positive integers $a_1, \dots, a_n$. The construction is closely related to the orbifold charts on weighted projective spaces defined in §3.50.

For simplicity we assume that we are in characteristic zero.

For each i between 1 and n, define a morphism $p_i : \mathbb{A}^n \to \mathbb{A}^n$ by

$$x_j = x'_j (x'_i)^{a_j} \quad \text{if} \quad j \neq i \quad \text{and} \quad x_i = (x'_i)^{a_i}.$$

The map p_i is birational if and only if $a_i = 1$; more generally, it has degree a_i. Note that the map p_i is well defined on the orbits of the $\mathbb{Z}_{a_i}$-action

$$(x'_1, \dots, x'_n) \mapsto (\epsilon^{-a_1} x'_1, \dots, \epsilon^{-a_{i-1}} x'_{i-1}, \epsilon x'_i, \epsilon^{-a_{i+1}} x'_{i+1}, \dots, \epsilon^{-a_n} x'_n),$$

where $\mathbb{Z}_{a_i}$ is a cyclic group of order a_i generated by a primitive a_ith root of unity ϵ. Therefore, p_i descends to a birational morphism π_i from the quotient variety:

$$\mathbb{A}^n / \mathbb{Z}_{a_i} \xrightarrow{\pi_i} \mathbb{A}^n.$$

These maps patch together to give a birational projective morphism

$$\pi : B_{(a_1, \dots, a_n)} \mathbb{A}^n \to \mathbb{A}^n;$$

this is precisely the *weighted blowup* of $\mathbb{A}^n$ with weights $a_1, \dots, a_n$. The exceptional set of π is a reduced irreducible divisor isomorphic to the weighted projective space $\mathbb{P}^{n-1}(a_1, a_2, \dots, a_n)$.

Points in the chart $\mathbb{A}^n / \mathbb{Z}_{a_i}$ can be described, somewhat deceptively, by their *orbifold coordinates* $x'_1, \dots, x'_n$. Some caution is in order since these coordinates are defined only up to the $\mathbb{Z}_{a_i}$ action on $\mathbb{A}^n$. The exceptional divisor is defined by the vanishing of the coordinate x'_i in this chart.

Discrepancy computations on weighted blowups are manageable because locally weighted blowups look like affine space up to an unramified cover. More precisely, the quotient maps

$$\mathbb{A}^n \to \mathbb{A}^n / \mathbb{Z}_{a_i} \tag{6.38.1}$$

giving local orbifold coordinates are unramified outside a set of codimension two, so by Exercise 6.26, many discrepancy computations on the singular variety $\mathbb{A}^n / \mathbb{Z}_{a_i}$ can be pulled back to $\mathbb{A}^n$. To see that the quotient map (6.38.1) is unramified in codimension one, note that it is ramified precisely at the fixed points of the $\mathbb{Z}_{a_i}$-action on $\mathbb{A}^n$. Now let ϵ be a generator for $\mathbb{Z}_{a_i}$. The point $(x'_1, \dots, x'_n)$ is a fixed point of ϵ^d if and only if

$$x'_i = \epsilon^d x'_i \quad \text{and} \quad x'_j = \epsilon^{-d a_j} x'_j \quad \text{for } j \neq i.$$

This is equivalent to

$$x_i' = 0 \quad \text{and} \quad x_j' = 0 \quad \text{whenever } a_i \text{ does not divide } da_j.$$

But if a_i divides da_j for every $j \neq i$, we contradict the assumption that the a_i are relatively prime. Thus the fixed point set of the $\mathbb{Z}_{a_i}$-action has codimension at least two.

The use of weighted blowups is illustrated by the proof of the next proposition. This also turns out to be the key step in the proof of Theorem 6.40.

PROPOSITION 6.39. *The log canonical threshold of the plane curve C defined by the vanishing of $x^b + y^a$ is $\frac{1}{a} + \frac{1}{b}$, whenever $a, b \geq 2$.*

PROOF. For notational clarity, we assume that a and b are relatively prime, which corresponds to the case where the curve is irreducible. The general case is proved in exactly the same way, but we use the weights $\frac{a}{m}$ and $\frac{b}{m}$, where m is the greatest common divisor of a and b, instead of the weights a and b, for the weighted blowing up map π.

Consider the weighted blowup

$$\pi : B = B_{(a,b)}\mathbb{A}^2 \to \mathbb{A}^2,$$

with exceptional fiber $E = \mathbb{P}^1(a, b)$. We compute the log canonical threshold of C by considering its birational transform C' on B.

We first claim that

$$K_B = \pi^* K_{\mathbb{A}^2} + (a + b - 1)E \quad \text{and} \quad C' = \pi^* C - abE. \qquad (6.39.1)$$

To check this, we compute locally in one of the affine charts of B. Consider the chart $\mathbb{A}^2/\mathbb{Z}_b$, with orbifold coordinates x', y' (defined only up to the $\mathbb{Z}_b$-action $x' \mapsto \epsilon^{-a}x'$, $y' \mapsto \epsilon y'$). The blowing up map is given by

$$\pi_2 : \mathbb{A}^2/\mathbb{Z}_b \to \mathbb{A}^2$$
$$(x', y') \mapsto (x'y'^a, y'^b) = (x, y).$$

We therefore compute directly that

$$\pi_2^*(dx \wedge dy) = by'^{a+b-1} dx' \wedge dy' \quad \text{and} \quad \pi_2^*(x^b + y^a) = y'^{ab}(x'^b + 1).$$

Because the exceptional divisor E is defined by the vanishing of y' in this chart, formula (6.39.1) is proved.

Fix any positive rational number c. From formula (6.39.1) we have

$$(K_B + cC') - \pi^*(K_{\mathbb{A}^2} + cC') = (a + b - 1 - cab)E.$$

If $(\mathbb{A}^2, cC)$ is log canonical, then $a+b-1-cab \geq -1$. It follows that $c \leq \frac{1}{a}+\frac{1}{b}$, and the log canonical threshold of C is bounded above by $\frac{1}{a}+\frac{1}{b}$.

To check that this bound is sharp, we must verify that $(\mathbb{A}^2, (\frac{1}{a}+\frac{1}{b})C)$ is log canonical. Using the formulas (6.39.1), we can write

$$K_B + E + (\tfrac{1}{a}+\tfrac{1}{b})C' \equiv \pi^*(K_{\mathbb{A}^2} + (\tfrac{1}{a}+\tfrac{1}{b})C).$$

Again, the very useful Exercise 6.15 implies that the pair $(\mathbb{A}^2, (\frac{1}{a}+\frac{1}{b})C)$ is log canonical if and only if the pair $(B, E+(\frac{1}{a}+\frac{1}{b})C')$ is. The advantage of working on B is that C' takes a very simple form there.

Now we exploit the fact that the local quotient maps $\mathbb{A}^2 \to \mathbb{A}^2/\mathbb{Z}_b$ and $\mathbb{A}^2 \to \mathbb{A}^2/\mathbb{Z}_a$ are unramified outside a set of codimension two. By Exercise 6.26, the pair $(B, E+(\frac{1}{a}+\frac{1}{b})C')$ is log canonical if and only if, on both of the charts, the pair $(\mathbb{A}^2, \tilde{E}+(\frac{1}{a}+\frac{1}{b})\tilde{C})$ is log canonical, where $\tilde{E}$ (respectively, $\tilde{C}$) denotes the pull-back of E (respectively, C) to $\mathbb{A}^2$ under the respective quotient maps.

Finally, we can use inversion of adjunction (Theorem 6.29) to reduce to a one dimensional problem. Indeed, by inversion of adjunction, the pair $(\mathbb{A}^2, \tilde{E}+(\frac{1}{a}+\frac{1}{b})\tilde{C})$ is log canonical if and only if the pair $(\tilde{E}, (\frac{1}{a}+\frac{1}{b})\tilde{C}|_{\tilde{E}})$ is log canonical. Note that on one of the charts, the equation of $\tilde{E}$ is $y'=0$ and the equation of $\tilde{C}$ is $(x')^b+1=0$ (on the other chart, the equations are $x''=0$ and $(y'')^a+1=0$ respectively). Thus $\tilde{C}|_{\tilde{E}}$ is a collection of b distinct points on one chart and a on the other. (On C we have only one point. The map $\tilde{C} \to C$ has degree b in one chart and degree a in the other, thus the differing number of points.) In any case, it follows that $(\tilde{E}, (\frac{1}{a}+\frac{1}{b})\tilde{C}|_{\tilde{E}})$ is log canonical since $\frac{1}{a}+\frac{1}{b} \leq 1$. □

Proposition 6.39 is a special case of a general result valid for every curve on a surface. Indeed, the above computation shows that the log canonical threshold of the divisor defined by $f = x^b + y^a$ can be expressed as

$$\frac{w(x)+w(y)}{\mathrm{mult}_w(f)}$$

where $w(x)$ denotes the weight of x (namely a), $w(y)$ denotes the weight of y (namely b), and $\mathrm{mult}_w(f)$ denotes the total weight of f (namely ab). In general, given weights $w(x)$ and $w(y)$ for a local coordinate system x, y, we define $\mathrm{mult}_w(f)$ to be the weight of the lowest weight term of f expressed as a power series in the coordinates x and y. The next theorem generalizes the formula of Proposition 6.39 to any curve on a smooth surface.

THEOREM 6.40 (Varčenko, 1976). *Let C be a curve on a smooth surface S. Then the log canonical threshold of C at P is equal to*

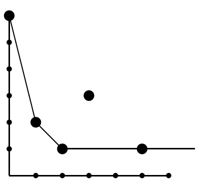
$$\inf_{x,y,w} \frac{w(x)+w(y)}{\mathrm{mult}_w(f)},$$

where the infimum runs over all local coordinate systems (x, y) for S at P and over all choices of weights $w(x)$ and $w(y)$ (positive integers), and where $f = 0$ is the equation of the curve C in the coordinates x, y.

The infimum is a minimum for analytic coordinate systems.

There is a nice geometric/combinatorial interpretation of the number $\frac{w(x)+w(y)}{\mathrm{mult}_w(f)}$ given by the Newton polygon, which we now explain.

6.41. NEWTON POLYGON. Consider a polynomial or power series $f = \sum a_{ij}x^i y^j$ in two variables. In a coordinate plane, put a dot at the point (i, j) if $a_{ij} \neq 0$. Add a horizontal line extending to the right of the lowest point and a vertical line extending upward from the leftmost point (these may be on the coordinate axes). The *Newton polygon* of f (with respect to the coordinates x and y) is the boundary of the convex hull of the resulting (infinite) figure.



The Newton polygon of
$x^5y + x^3y^3 + x^2y + xy^2 + y^6$.

The next lemma gives an interpretation of the quantity in Theorem 6.40 in terms of Newton polygons.

LEMMA 6.42. *Fix a power series f in coordinates x and y. Then as $w(x)$ and $w(y)$ range over all possible choices for (positive) weights, the function*

$$\frac{w(x)+w(y)}{\mathrm{mult}_w(f)}$$

is minimized by the choice such that

1. *the entire Newton polygon is contained in the halfplane $w(x)i + w(y)j \geq \mathrm{mult}_w f$, and*
2. *the line $w(x)i + w(y)j = \mathrm{mult}_w f$ contains the point $CP(f)$ where the Newton polygon intersects the diagonal line $j = i$.*

Thus we have three possibilities.

3. *$CP(f)$ is an interior point of an edge of the Newton polygon which is neither vertical nor horizontal. Then there is a unique minimizing choice of relatively prime integral weights.*
4. *$CP(f)$ is a vertex of the Newton polygon, not on a vertical or horizontal edge. Then we can choose relatively prime integral weights corresponding to the unique edge which also contains a vertex (i', j') with $i' < j'$.*
5. *$CP(f)$ is on a vertical or horizontal edge of the Newton polygon. There is no minimizing choice if $CP(f)$ is not a vertex.*

PROOF. First note that given any weights $w(x)$, $w(y)$, the weight of a monomial $x^i y^j$ is $w(x)i + w(y)j$. Therefore, the line

$$w(x) \cdot i + w(y) \cdot j = d$$

lies below the Newton polygon if and only if $d \leq \mathrm{mult}_w(f)$. Thus, as we vary the weights w, the lines

$$w(x) \cdot i + w(y) \cdot j = \mathrm{mult}_w(f)$$

all touch the Newton polygon in just one point (or coincide with one of its edges). The value $\frac{\mathrm{mult}_w(f)}{w(x)+w(y)}$ is the coordinate of the intersection point of the lines

$$w(x) \cdot i + w(y) \cdot j = \mathrm{mult}_w(f) \quad \text{and} \quad i = j.$$

Clearly $\frac{\mathrm{mult}_w(f)}{w(x)+w(y)}$ is maximized when the weights are such that the line $w(x) \cdot i + w(y) \cdot j = \mathrm{mult}_w(f)$ corresponds to the edge of the Newton polygon intersecting the $i = j$ line. Taking reciprocals, the conditions of (1) and (2) are proved. Finally, it is easy to verify the situations for the three possible positions of the point $CP(f)$. □

PROOF OF THEOREM 6.40. We first show that the log canonical threshold is bounded above by the stated infimum. Fix coordinates x and y at P, and let a and b be any relatively prime integral weights for x and y respectively. Consider the weighted (a, b)-blowup

$$\pi : S' \to S$$

with respect to the coordinates x and y. Arguing as in the proof of Theorem 6.39, we have that

$$K_{S'} = \pi^* K_S + (a + b - 1)E \qquad \text{and} \qquad C' = \pi^* C - dE,$$

where d is the multiplicity of f with respect to the weights w. So for any rational number c we have

$$(K_{S'} + cC') - \pi^*(K_S + cC) = (a + b - 1 - dc)E. \tag{6.40.1}$$

Thus if the pair (S, cC) is log canonical, then $(a + b - 1 - dc) \geq -1$, or equivalently $c \leq \frac{a+b}{d}$. As we range over all choices of coordinates and all choices of weights, this shows that the log canonical threshold is at most $\inf \frac{w(x)+w(y)}{\mathrm{mult}_w(f)}$.

It remains to prove the reverse inequality. Fix coordinates x and y.

Case (5) of Lemma 6.42 is settled by Exercise 6.43.

Otherwise we are in Cases (3) or (4) and, for these coordinates, there is a choice of relatively prime integral weights, say $w(x) = a$ and $w(y) = b$ minimizing the function $\frac{w(x)+w(y)}{\mathrm{mult}_w(f)}$, as follows from Lemma 6.42. We show that for this choice, either $\frac{w(x)+w(y)}{\mathrm{mult}_w(f)}$ is equal to the log canonical threshold, or else we can change coordinates so as to get a smaller value of $\frac{w(x)+w(y)}{\mathrm{mult}_w(f)}$. Repeating this procedure, we eventually converge to the log canonical threshold of C.

Taking $c = \frac{d}{a+b}$ in formula (6.40.1), we see that

$$K'_S + E + \frac{a+b}{d} C' = \pi^* \left(K_S + \frac{a+b}{d} C \right).$$

Thus by Exercise 6.15, the pair $(S, \frac{a+b}{d} C)$ is log canonical if and only if the pair $(S', E + \frac{a+b}{d} C')$ is log canonical. To check whether $(S', E + \frac{a+b}{d} C')$ is log canonical, we check locally on the orbifold cover of S' as we did in the proof of Proposition 6.39.

In one chart, we pull back to $\mathbb{A}^2$ under the map

$$x' \mapsto x'(y')^a = x \qquad y' \mapsto (y')^b$$

and denote the resulting pull-back pair by $(\mathbb{A}^2, \tilde{E} + \frac{a+b}{d} \tilde{C})$. By inversion of adjunction, the pair $(\mathbb{A}^2, \tilde{E} + \frac{a+b}{d} \tilde{C})$ is log canonical if and only if the pair $(\tilde{E}, \frac{a+b}{d} \tilde{C}|_{\tilde{E}})$ is. In this chart, the divisor $\tilde{E}$ is defined by the vanishing of y' and $\tilde{C}$ is defined by vanishing of the polynomial $\frac{f(x'(y')^a, (y')^b)}{(y')^d}$. Thus $(\tilde{E}, \frac{a+b}{d} \tilde{C}|_{\tilde{E}})$ is log canonical if and only if the roots of the polynomial $p(x')$ obtained by setting $y' = 0$ in the polynomial $\frac{f(x'(y')^a, (y')^b)}{(y')^d}$ occur with multiplicities at most $\frac{d}{a+b}$.

Now, in the event that the roots of $p(x')$ do occur with multiplicities less than or equal to $\frac{d}{a+b}$, the pair $(\tilde{E}, \frac{a+b}{d} \tilde{C}|_{\tilde{E}})$ and hence the pair $(S, \frac{a+b}{d} C)$ is log canonical. In this case, the log canonical threshold of C is exactly $\frac{a+b}{d}$ and the proof is complete. Thus, we need only consider the case where $p(x')$ admits a root α occuring with multiplicity $e > \frac{d}{a+b}$.

Let $f_d(x, y)$ be the lowest weight part of f (that is, the sum of the terms of weight d).

If $\alpha = 0$ then x^e divides $f_d(x, y)$, and the largest y-power in $f_d(x, y)$ is at most

$$\frac{1}{b}(d - ae) < \frac{1}{b}\left(d - a\frac{d}{a+b}\right) = \frac{d}{a+b} < e.$$

Thus the monomials corresponding to $f_d(x, y)$ all lie below the diagonal $i = j$, a contradiction. Thus $\alpha \neq 0$.

By Exercise 6.44, $(x' - \alpha)^e$ divides $p(x')$ if and only if $(x^b - \alpha^b y^a)^e$ divides $f_d(x, y)$. Since the weight of $(x^b - \alpha^b y^a)^e$ is abe and the weight of f_d is d, we have $abe \leq d$. This means that $e \leq \frac{d}{ab}$, whence $e \leq \frac{d}{a+b}$ if both a and b are greater than one. So we can assume that either a or b is one – say that $b = 1$ for the sake of argument – and that $f_d(x, y)$ is divisible by $(x - \alpha y^a)^e$ for some $e \geq \frac{d}{a+b}$.

Our strategy now is to change coordinates (and weights) so as to make the new value of $\frac{w(x)+w(y)}{\mathrm{mult}_w f}$ closer to the log canonical threshold. We set

$$x_1 = x - \alpha y^a \qquad \text{and} \qquad y_1 = y,$$

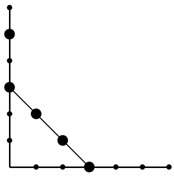
and let $f^{(1)}(x_1, y_1) = 0$ be the equation for C in these new coordinates. If the weights $w(x_1) = a$ and $w(y_1) = b$ are minimizing for the function $\frac{w(x_1)+w(y_1)}{\mathrm{mult}_w f^{(1)}}$, then $ai + bj = d$ is the equation of the edge of the Newton polygon for $f^{(1)}$ intersecting the line $j = i$ (see Lemma 6.42). In this case, there is a monomial $x_1^i y_1^j$ appearing in $f^{(1)}$ satisfying $ai + bj = d$ and $i < j$ (or the Newton polygon has a vertical edge and the proof is complete by Exercise 6.43 below). But for these values of i and j, then, we have

$$i = \frac{ai + bi}{a+b} \leq \frac{ai + bj}{a+b} = \frac{d}{a+b}.$$

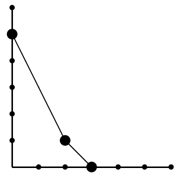
Note that this monomial also appears in $f_d^{(1)}$. But since $f_d^{(1)}$ is divisible by x_1^e, the x-exponent of each of its terms is at least e. This means that $i \geq e > \frac{d}{a+b}$, a contradiction.

Therefore, we may assume that the choice of the weights $w(x_1) = a$ and $w(y_1) = b$ is not optimal. Next we choose new weights $w(x_1) = a_1$ and $w(y_1) = b_1$ so as to minimize the function $\frac{w(x_1)+w(y_1)}{\mathrm{mult}_w f^{(1)}}$. Thinking about the shape of the new Newton polygon versus the old one, we see that necessarily $a_1 > a$ and $a_1 > b_1$. That is, the edge of the Newton polygon intersecting the line $j = i$ is now sloping more sharply down to the right.

The following example illustrates the effect on the Newton polygon under the coordinate change $x_1 = x + y$, $y_1 = y$.



The Newton polygon of $x^3 + x^2y - xy^2 - y^3 + y^5$, showing threshold $\leq 2/3$.



The Newton polygon of $x_1^3 - 2x_1^2y_1 + y_1^5$, showing threshold $\leq 3/5$.

We can now carry out the same argument in the new coordinates x_1 and y_1. This either proves the theorem or leads to another change of coordinates. Note that since $a_1 > b_1$, if a coordinate change is necessary, then again $b_1 = 1$ and this coordinate change has the same form as before. Repeating, we eventually prove the theorem or we get an infinite sequence of changes of the x-coordinate form a system

$$x_1 = x - \alpha_0 y, x_2 = x_1 - \alpha_1 y^{a_1}, x_3 = x_2 - \alpha_2 y^{a_2}, \ldots$$

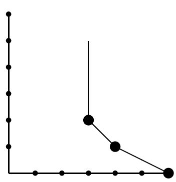
and $a_1 < a_2 < \cdots$. Thus we can put all these together into one coordinate change by formal power series

$$x_\infty = x - \sum_i \alpha_i y^{a_i}, \qquad y_\infty = y.$$

In this coordinate system the Newton polygon has a vertical edge. The correctness of the formula in this case is left as Exercise 6.43 below.

Using the methods of Arnold *et al.* (1985, I.I.6) it is not hard to see that one can make a convergent analytic coordinate change as well. □

Exercise 6.43. Assume that the Newton polygon of C has a vertical edge $i = i_0$ which intersects the line $i = j$. Prove that the log canonical threshold of C at P is $\frac{1}{i_0}$.



The Newton polygon of $x^6 + x^4y + x^3y^2$, showing that the log canonical threshold is $1/3$.

Exercise 6.44. Let k be an algebraically closed field and a, b relatively prime weights.

1. The irreducible weighted homogeneous polynomials are x, y and $\beta x^b - \alpha y^a$ for $\alpha, \beta \neq 0$.
2. Every weighted homogeneous polynomial is the product of irreducibles.

EXERCISE 6.45. Consider a divisor D in $\mathbb{C}^3$ defined by the vanishing of a polynomial

$$g_m = (x^2 + y^2 + z^2)^2 + ax^m + by^m + cz^m.$$

where a, b, and c be general complex numbers and $m \geq 5$. Show that the log canonical threshold of D is $\frac{1}{2} + \frac{1}{m}$.

Show that trying to compute the log canonical threshold using weights as in Theorem 6.40 gives only an upper bound of $\frac{3}{4}$. To be precise, let x', y', z' be any coordinate system, $g'_m(x', y', z') := g_m(x, y, z)$ and $w(x'), w(y'), w(z')$ weights, then

$$\frac{w(x') + w(y') + w(z')}{\operatorname{mult}_w g'_m(x', y', z')} \geq \frac{3}{4}.$$

6.6 Zero-dimensional maximal centers on threefolds

Finally, we are in a position to prove Theorem 5.20. This also concludes the proof that no smooth quartic threefold is rational.

We first recall the statement of Theorem 5.20: *Let H be a mobile linear system on a smooth threefold X and let P be a zero-dimensional maximal center of $(X, \frac{1}{m}H)$. Then for general members H_1 and H_2 of H and general smooth surface S through P, the local intersection number $(H_1 \cdot H_2 \cdot S)_P$ is defined and greater than $4m^2$.*

Our proof strategy is to use inversion of adjunction to reduce to a statement about curves on surfaces. This surface statement is then handled by the following corollary of Theorem 6.40.

COROLLARY 6.46. *Let C be a mobile linear system on a smooth surface S, and let P be a point on S. If the pair $(S, \frac{1}{m}C)$ is not log canonical at P, then the local intersection multiplicity $(C_1 \cdot C_2)_P$ is greater than $4m^2$, where C_1 and C_2 are general members of C.*

PROOF. As in Theorem 6.40, choose a local coordinate system (x, y) at P and weights $w(x), w(y)$ such that the log canonical threshold of C at P is

$\frac{w(x)+w(y)}{\text{mult}_w(f)}$. Since $(S, \frac{1}{m}C)$ is not log canonical at P, we know that

$$\frac{1}{m} > \frac{w(x)+w(y)}{\text{mult}_w(f)}.$$

On the other hand,

$$(C \cdot C')_P \geq \frac{\text{mult}_w(f)^2}{w(x) \cdot w(y)}$$

by Lemma 6.47 below, thus it is sufficient to prove that

$$\frac{\text{mult}_w(f)^2}{w(x) \cdot w(y)} \geq 4 \left(\frac{\text{mult}_w(f)}{w(x)+w(y)} \right)^2.$$

This is immediate, since it is equivalent to $(w(x) - w(y))^2 \geq 0$. □

LEMMA 6.47. *Let C and C' be two curves on a smooth surface having no common components and intersecting at a point P. Then*

$$(C \cdot C')_P \geq \frac{\text{mult}_w(f) \cdot \text{mult}_w(f')}{w(x) \cdot w(y)}$$

where x and y are local coordinates at P, f (respectively f') defines C (respectively C') in those coordinates, and $w(x), w(y)$ is any choice of weights.

PROOF. Consider the map $q : \mathbb{A}^2_{(u,v)} \to \mathbb{A}^2_{(x,y)}$ given by $x = u^{w(x)}$, $y = v^{w(y)}$. Then q has degree $w(x)w(y)$, hence

$$(q^*C \cdot q^*C')_P = w(x)w(y) \cdot (C \cdot C')_P.$$

On the other hand, q^*C (respectively q^*C') have multiplicity $\text{mult}_w(f)$ (respectively $\text{mult}_w(f')$), hence

$$(q^*C \cdot q^*C')_P \geq \text{mult}_w(f) \cdot \text{mult}_w(f').$$

Dividing by $w(x)w(y)$ gives the result. □

We are finally ready to prove Theorem 5.20. We begin with an easy lemma.

LEMMA 6.48. *Let Δ be a $\mathbb{Q}$-linear combination of linear systems on a smooth variety X. Let P be any point on X and let S be any hypersurface through P. Then*

$$a(E, X, \Delta) - 1 \geq a(E, X, S + \Delta)$$

for any divisor over X whose center on X is P.

PROOF. Consider any birational morphism $f : Y \to X$. Write

$$(K_Y - f^*K_X) + (f_*^{-1}S - f^*S) + (f_*^{-1}\Delta - f^*\Delta) = \sum e_i E_i \qquad (6.48.1)$$

where the E_i are all exceptional for f. Since S passes through P, every exceptional divisor lying over P must appear in f^*S with positive (integral) multiplicity. Since no exceptional divisor appears in $f_*^{-1}S$, we see that every exceptional divisor of $(f_*^{-1}S - f^*S)$ appears with coefficient at most -1. Thus

$$\operatorname{discrep}(X, \Delta) - 1 \geq \operatorname{discrep}(X, S + \Delta).$$

The proof is complete. □

REMARK 6.49. As usual, it is not really necessary to assume that S and X are smooth, as long as S is Cartier and we can make sense of the pull-backs of the appropriate systems in the proof.

PROOF OF THEOREM 5.20. Consider a mobile linear system H on a smooth threefold X. Suppose that P is a zero-dimensional maximal center of $(X, \frac{1}{m}H)$. By Lemma 6.48, this means that $(X, S + \frac{1}{m}H)$ is not log canonical in a neighborhood of P, where S is any smooth surface through P. By inversion of adjunction, then, the pair $(S, \frac{1}{m}H|_S)$ is not log canonical at P either.

For general members H_1 and H_2 of H, the local intersection number $(H_1 \cdot H_2 \cdot S)_P$ computed on X is the same as the local intersection number of the curves $C_1 = H_1|_S$ and $C_2 = H_2|_S$ at P on the surface S. By Corollary 6.46, therefore, this intersection number is greater than $4m^2$. This completes the proof. □

6.7 Appendix: proof of the connectedness theorem

We give a proof of the connectedness theorem, Theorem 6.32, which was used in the proof of inversion of adjunction. The proof is quite short but it uses higher direct images and a refined form of the Grauert–Riemenschneider vanishing theorem.

We begin our discussion with a far simpler result, which contains the key idea.

THEOREM 6.50. *Let $D = \sum d_i D_i$ be an effective $\mathbb{Q}$-divisor on a smooth variety X. Assume that $-(K_X + D)$ is ample and that the support of D has* *simple normal crossings. Set*

$$F = \sum_{d_i \geq 1} d_i D_i.$$

Then Supp F *is connected.*

6.51. To see the method first in a simpler case, we begin with assuming in addition that D is an integral divisor. In this case $F = D$ and we need to show that D is connected.

Our strategy is to show that $h^0(\mathcal{O}_D) = 1$, which implies that D is connected. Consider the exact sequence

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0.$$

Now $H^1(\mathcal{O}_X(-D)) = H^1(\mathcal{O}_X(K + (-K - D))) = 0$ by the vanishing theorem of Kodaira (Griffiths and Harris, 1978, 1.2). The long exact sequence of cohomology then implies that $h^0(\mathcal{O}_D) = 1$.

The general case uses the vanishing theorem of Kawamata and Viehweg, a generalization of the Kodaira vanishing theorem where the idea is to regard a $\mathbb{Q}$-divisor as a "small" perturbation of its integer part. (See Kollár and Mori, 1998, 2.5) for a relatively simple proof.)

THEOREM 6.52 (Generalized Kodaira vanishing). *Let X be a smooth projective variety over $\mathbb{C}$. Let L be a Cartier divisor on X and assume that L is numerically equivalent to a $\mathbb{Q}$-divisor of the form $H + \sum a_i D_i$ where H is ample, the support of $\sum D_i$ has simple normal crossings and* $0 \leq a_i < 1$. *Then*

$$H^i(X, \mathcal{O}_X(K_X + L)) = 0 \quad \textit{for} \quad i > 0.$$

We also need some notation to work with nonintegral divisors.

DEFINITION 6.53. The *round down* of a real number d is the largest integer less than or equal to d. Likewise, *round up* is the smallest integer $\geq d$. The round down (respectively, round up) of d is denoted by $\lfloor d \rfloor$ (respectively, $\lceil d \rceil$). The *fractional part* of d is $\{d\} := d - \lfloor d \rfloor$ If $D = \sum d_i D_i$ is a divisor with real coefficients and the D_i are distinct prime divisors, then we define the *round down* of D as $\lfloor D \rfloor := \sum \lfloor d_i \rfloor D_i$, the *round up* of D as $\lceil D \rceil := \sum \lceil d_i \rceil D_i$ and the *fractional part* of D as $\{D\} := \sum \{d_i\} D_i$.

PROOF OF THEOREM 6.50. Using the notation of Definition 6.53, write

$$D = \lfloor D \rfloor + \{D\}.$$

Note that $F = \operatorname{Supp}\lfloor D \rfloor$. As before, consider the exact sequence

$$0 \to \mathcal{O}_X(-\lfloor D \rfloor) \to \mathcal{O}_X \to \mathcal{O}_{\lfloor D \rfloor} \to 0.$$

It is now easy to conclude as before:

$$\begin{aligned} H^1(\mathcal{O}_X(-\lfloor D \rfloor)) &= H^1(\mathcal{O}_X(K + (-K - \lfloor D \rfloor))) \\ &= H^1(\mathcal{O}_X(K - (K + D) + \{D\})) = 0 \end{aligned}$$

by the vanishing Theorem 6.52. □

6.54. To further warm up to the proof of the connectedness theorem, first we look at the case when all occurring divisors are Cartier. Thus assume that K_X and D_X are Cartier divisors. Then D is also Cartier and $M = 0$. Write $D = F - A$ where $A = -\sum_{d_i \le 0} d_i D_i$. Note that both F and A are effective Cartier divisors and they have no irreducible components in common. Moreover, A is contained in the exceptional locus of g. Consider the exact sequence

$$0 \to \mathcal{O}_Y(A - F) \to \mathcal{O}_Y(A) \to \mathcal{O}_F(A|_F) \to 0.$$

Applying g_* we obtain the exact sequence

$$g_*\mathcal{O}_Y(A) \to g_*\mathcal{O}_F(A|_F) \to R^1 g_*\mathcal{O}_Y(A - F).$$

The conclusion of the theorem is local on X, so by working in a small neighborhood of a point of X we may assume that $K_X + D_X \sim 0$. This implies that $A - F \sim K_Y$, thus $R^1 g_*\mathcal{O}_Y(A - F) = 0$ by Grauert–Riemenschneider vanishing, Theorem 6.57.1. Note that $g_*\mathcal{O}_Y(A) = \mathcal{O}_X$ since A is g-exceptional and effective. Thus we have a surjection

$$\mathcal{O}_X \twoheadrightarrow g_*\mathcal{O}_F(A|_F).$$

Assume that F has at least two connected components $F = F_1 \cup F_2$ in a neighborhood of $g^{-1}(x)$ for some $x \in X$. Then

$$g_*\mathcal{O}_F(A|_F)_{(x)} \cong g_*\mathcal{O}_{F_1}(A|_{F_1})_{(x)} \oplus g_*\mathcal{O}_{F_2}(A|_{F_2})_{(x)},$$

and neither of these summands is zero. Thus $g_*\mathcal{O}_F(A|_F)_{(x)}$ cannot be the quotient of the cyclic module $\mathcal{O}_{x,X}$.

The general case of the connectedness theorem is very similar but we have to deal appropriately with the fractional coefficients of the divisor D.

Proof of Theorem 6.32. We again write D as the difference of two effective divisors without common components $D = G - A$. Note that G is made up of F and other divisors whose coefficient in D is between 0 and 1. In particular, $\lfloor G \rfloor = \lfloor F \rfloor$ and $\operatorname{Supp}\lfloor G \rfloor = \operatorname{Supp} F$.

If D_i is an irreducible component of A then D_i is g-exceptional thus $\lceil A\rceil$ is g-exceptional and effective. Applying g_* to the exact sequence

$$0 \to \mathcal{O}_Y(\lceil A\rceil - \lfloor G\rfloor) \to \mathcal{O}_Y\big(\lceil A\rceil|_{\lfloor G\rfloor}\big) \to \mathcal{O}_{\lfloor G\rfloor}(\lceil A\rceil) \to 0,$$

we obtain

$$g_*\mathcal{O}_Y(\lceil A\rceil) \to g_*\mathcal{O}_{\lfloor G\rfloor}\big(\lceil A\rceil|_{\lfloor G\rfloor}\big) \to R^1 g_*\mathcal{O}_Y(\lceil A\rceil - \lfloor G\rfloor).$$

We need to work on the divisor $\lceil A\rceil - \lfloor G\rfloor$ to see that vanishing applies. Observe that

$$\begin{aligned}\lceil A\rceil - \lfloor G\rfloor &= A + \{-A\} - G + \{G\}\\ &= -D + \{-A\} + \{G\}\\ &= K_Y + M - (K_Y + D + M) + \{-A\} + \{G\}\\ &\equiv K_Y + (M - g^*(K_X + D_X)) + \{-A\} + \{G\}.\end{aligned}$$

M is a sum of free linear systems and $-g^*(K_X + D_X)$ is pulled back from X, so $M - g^*(K_X + D_X)$ has non-negative degree on any curve contracted by g. $\{-A\} + \{G\}$ has coefficients between 0 and 1 and its support is a simple normal crossing divisor. Therefore $R^1 g_*\mathcal{O}_Y(\lceil -A\rceil - \lfloor G\rfloor) = 0$ by Theorem 6.57.2. Thus we again have a surjection

$$\mathcal{O}_X \twoheadrightarrow g_*\mathcal{O}_{\lfloor G\rfloor}\big(\lceil A\rceil|_{\lfloor G\rfloor}\big),$$

and we obtain connectedness as before. □

REMARK 6.55. In the last step, we have identified $g_*\mathcal{O}_Y(\lceil A\rceil) = \mathcal{O}_X$. In doing so, we have used Exercise 6.56 below. For this, it is crucial that $\lceil A\rceil$ is g-exceptional and effective, that is, ultimately, that D_X is effective. Inversion of adjunction does not work if D_X is not effective. For instance, consider the pair $(\mathbb{A}^2, L_1 + 2L_2 - L_3)$, where L_1 is the y-axis, L_2 is the x-axis and L_3 is the line given by $y = x$. Then this pair is not log canonical but its restriction to the line L_1 is.

EXERCISE 6.56. Let $f : Z \to X$ be a proper birational morphism between normal varieties. Let $E = \sum k_i E_i$ be an integral Weil divisor on Z, where all E_i are f-exceptional. Then $f_*\mathcal{O}_Z(\sum k_i E_i) = \mathcal{O}_X$ if and only if all $k_i \geq 0$.

In the proof of the connectedness theorem, we used the Grauert–Riemenschneider vanishing theorem, as well as a more recent refinement:

THEOREM 6.57. *Let $g : Y \to X$ be a proper and birational morphism between varieties over $\mathbb{C}$ with Y smooth. Then*

1. $R^i g_* \omega_Y = 0$ *for* $i > 0$.
2. *(Generalized Grauert–Riemenschneider vanishing) Let* L *be a Cartier divisor on* Y *and assume that* L *is numerically equivalent to a* $\mathbb{Q}$*-divisor of the form* $M + \sum a_i D_i$ *where* $\sum D_i$ *has simple normal crossings,* $0 \le a_i < 1$ *for every* i *and* M *has non-negative degree on any curve contracted by* g. *Then*

$$R^i g_*(\mathcal{O}_Y(K_Y + L)) = 0 \quad \textit{for} \quad i > 0.$$

The first statement above is the classical Grauert–Riemenschneider vanishing theorem; see Grauert and Riemenschneider (1970); the general case is nowadays viewed as another variant of the generalized Kodaira vanishing 6.52. See Kollár and Mori (1998, 2.68) for a relatively simple proof of Theorem 6.57.

7
Solutions to exercises

7.1 Exercises in Chapter 1

SOLUTION TO 1.8.

That (1) implies (2) is clear. To see the converse, let

$$f = \sum a_I x^I \in k'[x_1, \ldots, x_n]$$

be a polynomial vanishing on X. For $\sigma \in \mathrm{Gal}(k'/k)$, set

$$f^\sigma := \sum \sigma^{-1}(a_I)x^I.$$

(Using the inverse of σ fits better into the general framework, see §3.28.) If X is Galois invariant, then f^σ also vanishes on X. Let $E_1(f), \ldots, E_d(f) \in k[x_1, \ldots, x_n]$ denote the elementary symmetric polynomials of $\{f^\sigma : \sigma \in \mathrm{Gal}(k'/k)\}$. Note that $E_1(f), \ldots, E_d(f)$ vanish on X and are defined over k. So if f_j runs through a set of defining polynomials for X, then the resulting symmetric polynomials $E_i(f_j)$ give polynomials over k which vanish on X.

Conversely, if $E_1(f), \ldots, E_d(f)$ all vanish at a point P, that is, all the elementary symmetric functions of $\{f^\sigma(P) : \sigma \in \mathrm{Gal}(k'/k)\}$ are zero, then every $f^\sigma(P)$ is zero since they are the roots of the polynomial

$$z^d = z^d + \sum_{i=1}^{d} (-1)^i E_i(f)(P) z^{d-i}.$$

Hence the common zero set of the $E_i(f_j)$ is precisely X.

Note, however, that scheme-theoretically this process does not necessarily produce the right polynomials. For instance, if we define $X = \{\sqrt{2}, -\sqrt{2}\} \subset \mathbb{A}^1$ by the equation $\sqrt{2}(x^2 - 2) = 0$ then $E_1 = 0$ and $E_2 = 2(x^2 - 2)^2$. Thus the vanishing of E_1 and E_2 defines a non-reduced structure on X.

It turns out that Exercise 1.8 also holds scheme theoretically. This is discussed in Section 4 of Chapter 3. □

SOLUTION TO 1.9.

Let $f = \sum a_I x^I \in k'[C]$ be a polynomial which vanishes at P with multiplicity one. Then $f^{p^a} = \sum a_I^{p^a} x^I \in k[C]$ and it vanishes at P with multiplicity p^a. □

SOLUTION TO 1.12.

The following proof is due to E. Szabó.

Use induction on the dimension of Y. If Y has dimension one, rational maps are defined everywhere, and the result is obvious. If Y is a smooth variety with a k-point P, blow up P to get a variety $\tilde{Y}$. The blowup map $\tilde{Y} \to Y$ is defined over k, and the exceptional fiber, being isomorphic to a projective space $\mathbb{P}$, has lots of k-points. Any rational map $Y \overset{\phi}{\dashrightarrow} Y'$ defined over k determines a rational map $\tilde{Y} \overset{\tilde{\phi}}{\dashrightarrow} Y'$. Because $\tilde{Y}$ is smooth and Y' is projective, the locus of indeterminacy has codimension at least two. This means that $\tilde{\phi}$ restricts to a rational map of the exceptional fiber $\mathbb{P}$. Because this variety has smaller dimension, we are done by induction.

If Y is not smooth, Nishimura's lemma can fail. Indeed, let Y be the projective closure of the affine cone over a smooth projective variety X with no k-points. Then Y has exactly one k point, the vertex of the cone. Blowing up the vertex, we achieve a smooth projective variety $\tilde{Y}$ with no k-points, since the exceptional fiber is k-isomorphic to X. The rational map $Y \dashrightarrow \tilde{Y}$ gives the counterexample to Nishimura's lemma in the case where the source is not smooth. □

SOLUTION TO 1.13.

Let $q(x_0, \ldots, x_n) = 0$ be the equation of Q. The singular locus of Q is defined by the equations $q = 0$ and $\partial q/\partial x_i = 0$. Since q is a homogeneous quadric, $2q = \sum x_i \partial q/\partial x_i$. Thus if the characteristic of k is not two, the singular locus of q is also defined by the linear equations $\partial q/\partial x_i = 0$, thus it is a linear space. Changing coordinates, we may assume that it is $x_0 = \cdots = x_m = 0$, in which case q becomes a polynomial in only the variables $x_0, \ldots, x_m$. Now take Q' to be the quadric in $\mathbb{P}^m$ defined by the polynomial q considered in the variables $x_0, \ldots, x_m$, and observe that Q is a cone over Q'. Since the partial derivatives of Q' and of Q are the same, their common zero set is $x_0 = \cdots = x_m = 0$. It follows that Q' is smooth, and also that if $(a_0 : \cdots : a_m)$ is a (smooth) point of Q', then $(a_0 : \cdots : a_m : a_{m+1} : \cdots : a_n)$ is a smooth point of Q. □

SOLUTION TO 1.14. This is the well-known standard form for a real quadratic form; see, for example, Artin (1991, p 245). The singular locus of Q is precisely the set where the coordinates $x_1, \ldots x_{p+q}$ vanish. If $q > 0$, then the point whose coordinates are $x_1 = x_{p+1} = 1$ and all other x_i equal to zero is a smooth real point of Q. By Theorem 1.11, it follows that Q is rational. However, if $q = 0$, then all real points of Q are singular and so Q is not rational over $\mathbb{R}$. □

SOLUTION TO 1.20.

Using induction, it is enough to find an extension of degree p for any prime p.

If k is algebraically closed, and K is strictly larger than k, then K contains an element x transcendental over k. Since K is finitely generated, not all of the fields

$$k(x) \subset k\left(x^{p^{-1}}\right) \subset k\left(x^{p^{-2}}\right) \subset k\left(x^{p^{-3}}\right) \subset \cdots$$

are contained in K. Consider the maximal value a such that $x^{p^{-a}}$ is contained in K. Then $K(x^{p^{-a-1}})$ is a degree p extension of K. If p is not the characteristic of K, it is separable as well.

If p is the characteristic of K, then we instead use the sequence of fields

$$k(x_1) \subset k(x_2) \subset k(x_3) \subset \cdots$$

where $x_1 = x$ and the x_i are defined recursively by $x_{i+1}^p + x_{i+1} + x_i = 0$.

The case when K is a prime field is equally easy. □

SOLUTION TO 1.21.

By Fermat's little theorem, $a^{p-1} = 1$ for any nonzero $a \in \mathbb{F}_p$. Thus $x_1^{p-1} + \cdots + x_{p-1}^{p-1}$ can not be zero unless all the x_i are zero. This hypersurface is smooth since the derivatives can not all simultaneously vanish. □

SOLUTION TO 1.24.

Everything is easy except maybe part (5). This follows once we observe that the Frobenius map can be defined invariantly. Indeed, on an affine cover, we can define the Frobenius map as the one dual to the qth power map on functions: for any k-algebra R, where the cardinality of k is $q = p^e$, we have a k-linear map $R \to R$ sending r to r^q. This trivially patches together to give a well defined global morphism agreeing with the one we defined in the Exercise. □

SOLUTION TO 1.26.

Fix a finite ground field k of cardinality q, and let Δ and Γ denote the diagonal and the graph of the Frobenius map, respectively, for projective n-space over k.

1. Prove that any d-dimensional subvariety of $\mathbb{P}^n \times \mathbb{P}^n$ is rationally equivalent to a sum $\sum a_i[L^i \times L^{d-i}]$, where L^i denotes an i-dimensional linear subspace of $\mathbb{P}^n$. (The topological version for $\mathbb{CP}^n$ follows from the Küneth

formula. Over arbitrary fields, it is easiest to prove this by pushing cycles around with various coordinate-wise actions of the multiplicative group $GL(1)$.)

2. Prove that $\Delta \sim \sum_{i=0}^{n}[L^i \times L^{n-i}]$ and $\Gamma \sim \sum_{i=0}^{n} q^{n-i}[L^i \times L^{n-i}]$.
3. Conclude that $(\Delta \cdot \Gamma) = 1 + q + \cdots + q^n$.
4. Check that the intersection of Δ and Γ is transversal.
5. Conclude that if X is a variety over k such that $X_{\bar{k}} \cong \mathbb{P}^n$, then the cardinality of the set of k-points of X is $1 + q + \cdots + q^n$. □

Solution to 1.29.

Assume that $(x_0 : x_1 : x_2 : x_3)$ is a rational solution. We may assume that the x_i are integers and they are relatively prime. Reducing modulo p we get that $f(x_0, x_1) \equiv 0 \mod p$, thus $x_0, x_1 \equiv 0 \mod p$ by assumption. Thus we can rewrite our equation as

$$p[f(x_2, x_3) + p^2 f(x_0', x_1')] = 0,$$

where $x_0' = x_0/p$ and $x_1' = x_1/p$ are integers. We now get that $x_2, x_3 \equiv 0 \mod p$, contradicting the relatively prime assumption. □

Solution to 1.31.

There are isomorphic open subsets $X^0 \subset X$ and $Y^0 \subset Y$, thus $X^0(\mathbb{R})$ and $Y^0(\mathbb{R})$ have the same number of connected components. However, $X(\mathbb{R})$ and $X^0(\mathbb{R})$ may have different number of connected components (this happens already for $X = \mathbb{P}^1$) so this is not very helpful.

To get a proof we note that there is a subset $Z \subset X$ of codimension at least two such that the birational map $\phi : X \dashrightarrow Y$ is defined on $X \setminus Z$. Then $X(\mathbb{R})$ and $X(\mathbb{R}) \setminus Z(\mathbb{R})$ have the same number of connected components (think of removing points from $\mathbb{R}^2$ or curves from $\mathbb{R}^3$).

Since $\phi(X(\mathbb{R}) \setminus Z(\mathbb{R}))$ is a dense subset of $Y(\mathbb{R})$, the number of connected components of $Y(\mathbb{R})$ is at most the number of connected components of $X(\mathbb{R})$. Reversing the roles of X and Y completes the proof.

One may be misled to think that the above proof shows that the number of connected components of the real points can go only down under rational maps. This is, however, not true. If $\phi : X \to Y$ is a morphism then the image of $X(\mathbb{R})$ need not be dense in $Y(\mathbb{R})$ (in the Euclidean topology) and we may completely miss some of the components of $Y(\mathbb{R})$. □

Solution to 1.34.

(1) Choose coordinates so that the disjoint n planes L_1 and L_2 are given by $\{X_0 = X_1 = \cdots = X_n = 0\}$ and by $\{X_{n+1} = X_{n+2} = \cdots = X_{2n+1} = 0\}$ respectively.

A cubic given by an equation of the form

$$\sum_{i\le n; j>n} a_{ijk} X_i X_j X_k$$

contains both planes. The generic member in this linear system of cubics in $\mathbb{P}^{2n+1}$ is smooth, because it admits the following smooth special member:

$$\sum_{i=0}^{n} \left(X_i^2 X_{i+n+1} + X_i X_{i+n+1}^2 \right). \tag{$*$}$$

This is easily checked by the Jacobian criterion (assuming the characteristic is not 3).

(2) To count the dimension of the linear system of cubics containing a fixed pair of disjoint planes L_1 and L_2, we count the number of monomials of degree 3 in $2n+2$ variables minus the number of those monomials involving only variables generating the ideal of L_1 or of L_2. The total is

$$\binom{2n+1+3}{3} - \binom{n+3}{3} - \binom{n+3}{3} = (n+1)^2(n+2),$$

so the dimension of the linear system is $(n+1)^2(n+2)-1$.

Finally, to find the dimension of the space of all cubics containing *any* pair of disjoint planes, we need to add the dimension of the space of such pairs of planes. As a generic pair of planes are disjoint, this is twice the dimension of the Grassmannian of n planes in $\mathbb{P}^{2n+1}$, or $2(n+1)^2$. So the dimension of the space of all smooth $2n$ dimensional cubic hypersurfaces containing a pair of disjoint n-planes is $(n+1)^2(n+4)-1$.

We still need to prove that this expected dimension is correct, that is, that a general cubic containing two n-planes contains only finitely many of them. For this it is enough to produce one (possibly singular) cubic which contains a pair of isolated n-planes. We check this for the above example (*).

Any plane close to $\{X_{n+1} = X_{n+2} = \cdots = X_{2n+1} = 0\}$ can be given by equations

$$X_{i+n+1} = \sum_{j=0}^{n} a_{ij} X_j \quad \text{for } i = 0, \ldots, n.$$

If this is contained in our cubic, then we have an equation

$$\sum_{i=0}^{n} \left[X_i \left(\sum_{j=0}^{n} a_{ij} X_j \right)^2 + \left(\sum_{j=0}^{n} a_{ij} X_j \right) X_i^2 \right] = 0.$$

Let us look first at the coefficient of X_i^3, namely $a_{ii}^2 + a_{ii} = 0$. Since we are looking at planes near the original one (corresponding to all $a_{ij} = 0$) this implies that $a_{ii} = 0$. Next look at the coefficient of $x_i^2 x_j$, namely $a_{ij} + a_{ji}^2 = 0$. By symmetry also $a_{ij}^2 + a_{ji} = 0$, and again as above we get that $a_{ij} = 0$.

We remark that our cubic contains a huge number of n-planes, for instance the 3^{n+1} examples given by equations $L_0 = \cdots = L_n = 0$ where each L_i is any one of the three possibilities $X_i, X_{i+n+1}, X_i + X_{i+n+1}$.

(3) A cubic hypersurface (and more generally, any hypersurface of degree at least 2) containing a linear subspace of dimension greater than n is never smooth. Indeed, choosing coordinates so that X contains the space defined by $\{X_0 = X_1 = \cdots = X_{n-1} = 0\}$,

$$X = \sum_{i=0}^{n-1} X_i f_i = 0,$$

where the f_i have degree two. Now X can not be smooth along the locus of points where $\{X_0 = X_1 = \cdots = X_{n-1} = 0\}$ and $\{f_0 = f_1 = \cdots = f_{n-1} = 0\}$, because the Zariski tangent space at these points is $2n+1$ dimensional. But because this locus is defined by only $2n$ equations, it must have non-empty intersection with X.

(4) $L_1 = (t : 0 : 1 : 0)$ and $L_2 = (0 : s : 0 : 1)$ is a pair of skew lines on T. The line connecting $P_1(t) = (t : 0 : 1 : 0)$ and $P_2(s) = (0 : s : 0 : 1)$ intersects T in one more point □

$$P(s,t) = (t\,(s^2 + t) : -s(s\,t^2 + 1) : s^2 + t : -(s\,t^2 + 1)).$$

Solution to 1.39.

It is easy to check that $(1 : 0 : 0 : 0)$ is the only $\mathbb{F}_2$ point satisfying the given equation. To see that this is the only such cubic surface, we use the following argument of Swinnerton-Dyer.

Given such a surface X, we can assume that its unique point P is $(1 : 0 : 0 : 0)$, and that the tangent plane there is given by the vanishing of x_0. Then the plane cubic curve obtained by intersecting X with this tangent plane has P as its only $\mathbb{F}_2$-point. Hence this cubic is necessarily the union of three lines intersecting at P, and coordinates can be chosen so that its equation is $x_2^3 + x_2^2 x_3 + x_3^3$. Thus our surface has equation

$$\begin{aligned} &x_1^3 + x_0^2 x_1 + x_1^2(\text{linear in } x_0, x_2, x_3) \\ &\quad + x_1 \left(\text{quadratic in } x_0, x_2, x_3 \text{ with no } x_0^2\right) + x_2^3 + x_2^2 x_3 + x_3^3. \end{aligned}$$

A coordinate change $x_2 \mapsto x_2 + x_1$ can eliminate $x_1^2 x_2$ and a coordinate change $x_3 \mapsto x_3 + x_1$ can eliminate $x_1^2 x_3$. By looking at points on $x_0 = x_3 = 0$ we must have a term $x_1 x_2^2$ and by looking at points on $x_2 = x_3 = 0$ we must have a term $x_0 x_1^2$.

Finally, by computing at the four points with only one coordinate 0 settles that $x_1 x_2 x_3$ is the only term involving 3 variables.

Thus we end up with

$$x_1^3 + x_1^2 x_0 + x_1 \left(x_0^2 + x_2^2 + x_3^2 + x_2 x_3\right) + x_2^3 + x_2^2 x_3 + x_3^3.$$

□

Solution to 1.40.

The following argument is due to Swinnerton-Dyer.

We may assume that the line is $x_2 = x_3 = 0$. Each plane through this line intersects the cubic in a residual conic, which has no points outside this line. So each residual conic is a pair of conjugate lines which meet on the line $x_2 = x_3 = 0$. This brings us to the equation

$$x_2 \left(x_0^2 + x_0 x_2 + x_2^2\right) + x_3 \left(x_1^2 + x_1 x_3 + x_3^2\right) + x_2 x_3 (\text{linear form}).$$

Finally, looking at $(1 : 0 : 1 : 1)$ and $(0 : 1 : 1 : 1)$ we get that, up to interchanging x_2, x_3, the equation is

$$x_2 \left(x_0^2 + x_0 x_2 + x_2^2\right) + x_3 \left(x_1^2 + x_1 x_3 + x_3^2\right) + x_2^2 x_3.$$

□

Solution to 1.42.

Recall that a linear system of plane cubics with up to seven assigned base points (including possibly one infinitely near another), no four collinear and no seven on a conic, has no unassigned base points (Hartshorne, 1977, p.399) Also recall that a linear system on a smooth surface is very ample if and only if imposing two more base points (including one infinitely near another) causes the dimension to drop by exactly two (Hartshorne, 1977, p.399)

Consider four general points $P_1, \ldots, P_4$ in $\mathbb{P}^2$ and let $\beta = |3H - P_1 - P_2 - P_3 - P_4|$ be the linear system of cubics in $\mathbb{P}^2$ passing through these points. Using the criterion above, we see that (the pull-back of) this linear system $\beta = |3H - E_1 - E_2 - E_3 - E_4|$, to the blowup of $\mathbb{P}^2$ at the four points is very ample. Using this linear system embed the blowup as a surface S in $\mathbb{P}^5$.

Claim: A generic projection of S to $\mathbb{P}^4$ is a surface S' with exactly one singular point.

First, for two general points P and Q on $\mathbb{P}^2$, consider the following two linear subsystems of β on S. Fixing defining equations $s_0, \ldots, s_5$ for generators of β, consider the linear subsystems whose defining equations satisfy:

$$\gamma := \left\{ s \in \beta \mid \frac{s(P)}{s_0(P)} = \frac{s(Q)}{s_0(Q)} \right\}$$
$$\alpha := \{ s \in \beta \mid s(P) = s(Q) = 0 \} .$$

Note that $\alpha \subset \gamma \subset \beta$, and the dimensions drop by exactly one with each successive condition imposed. The linear system γ determines a projection π of $\mathbb{P}^5$ to $\mathbb{P}^4$, sending S to, say, S'. By definition of γ, we have $(s_0(P) : s_1(P) : \cdots : s_5(P)) = (s_0(Q) : s_1(Q) : \cdots : s_5(Q))$, so that π sends P and Q to the same point of S'.

If π collapses some other point P' (possibly infinitely near P or Q) to $\pi(P) = \pi(Q)$, then we have that whenever $s(P) = s(Q) = 0$ for some $s \in \beta$, the vanishing $s(P')$ is forced as well. This makes P' an unassigned base point of α, contradicting the genericity assumption. Likewise, if π collapses two other points P' and Q' (Q' may be infinitely near P') to a single point of S', then the linear system $|3H - P_1 - P_2 - P_3 - P_4 - P - Q - P'|$ has an unassigned base point Q', again a contradiction, since six general points and the one special point P' impose no extra conditions.

Thus any projection of $S \subset \mathbb{P}^5$ to $\mathbb{P}^4$ that collapses two general points of S to a single point of S' can collapse *only* these two points to a single point. The argument will be complete once we have shown that a general projection $\mathbb{P}^5 \dashrightarrow \mathbb{P}^4$ cannot be one-to-one on S.

Consider the incidence correspondence

$$\Gamma = \{(P, Q, x) \mid P, Q, x \text{ collinear}\} \subset S \times S \times \mathbb{P}^5 .$$

Through any two distinct points of S, there is a unique line in $\mathbb{P}^5$, so the projection $\Gamma \to S \times S$ is surjective, and its fibers are all one-dimensional. It follows that Γ is irreducible and of dimension five.

Consider the other projection $\Gamma \to \mathbb{P}^5$. We know that if P and Q are collapsed to the same point of S' via π, then these are the only two points collapsed under π. This implies that the fiber over any point in the image of $\Gamma \to \mathbb{P}^5$ is simply the triple (P, Q, x), so the fibers are zero-dimensional. From this we conclude that $\Gamma \to \mathbb{P}^5$ is surjective. This means that a generic projection from any point in $\mathbb{P}^5$ cannot be one-to-one on S.

This implies that a generic projection of $S \subset \mathbb{P}^5$ to a hyperplane in $\mathbb{P}^5$ collapses precisely two points of S to a single point S' in the image (which is therefore a singular point of S'). This completes the proof. $\square$

The surface we described is called a del Pezzo surface in $\mathbb{P}^5$. The reader familiar with rational quartic scrolls in $\mathbb{P}^5$ (cf. Reid, 1997, Chapter 2 or Harris

1992, 8.17) should be able to prove that these scrolls also have the property that a generic projection to $\mathbb{P}^4$ produces exactly one double point. In 1901, Severi claimed that these are the only two examples of such surfaces with a single "apparent double point," as he called them (Severi, 1911, p.44). A modern version of his proof can be found in Russo (2000).

SOLUTION TO 1.44.

(1) The variety Y of $m \times n$ matrices of rank at most t is defined by the $t+1$-minors of an $m \times n$ matrix of indeterminates. It is easy to check that its dimension is $mn-(m-t)(n-t)$. Define the rational map

$$\begin{aligned} Y &\dashrightarrow \mathbb{A}^{mn-(m-t)(n-t)} \\ \lambda &\mapsto \{(\ldots, \lambda_{ij}, \ldots) \mid i \text{ or } j \le t\}, \end{aligned}$$

sending a matrix λ to the indicated string of its entries. This map is a birational equivalence because, whenever the upper left hand t-minor Δ of the $m \times n$ matrix λ is nonzero, we solve uniquely for each λ_{ij} with both i and j greater than t. Indeed, since all $(t+1)$-minors vanish, we can use the Laplace expansion to express any such λ_{ij} with $i, j > t$ as a polynomial in the λs from the first t columns and rows with denominators Δ. This proves that Y is a rational variety over any field.

The singular locus of Y is the subvariety of matrices of rank strictly less than t. Indeed, if an $m \times n$ matrix has rank t, than some t-minor is non-vanishing, so using the analogous map described above, we can map an open subset of Y containing this matrix isomorphically to an open subset of affine space. Thus every rank t matrix in Y is a smooth point. Conversely, if some matrix has rank less than t, all t-minors vanish, and considering the Laplace expansion of the $t+1$-minors defining Y, we see easily that the Jacobian matrix is zero in this case. This says that the tangent space at such a point has bigger than expected dimension and the point is a singular point of Y.

(2) Let X be the subvariety of $\mathbb{P}^n$ defined by the vanishing of the determinant of the $n \times n$ matrix $\mathbf{L}$ of general linear forms in $n+1$ variables. For each $n+1$ tuple $x = (x_0, x_1, \ldots, x_n)$, consider $\mathbf{L}(x)$ as a linear map $k^n \to k^n$. This defines a rational map

$$\begin{aligned} X \subset \mathbb{P}^n &\dashrightarrow \mathbb{P}^{n-1} \\ x = [x_0 : x_1 : \cdots : x_n] &\mapsto \{\text{kernel of the matrix } \mathbf{L}(x)\}. \end{aligned}$$

The genericity assumption guarantees that the matrix $\mathbf{L}$ has rank exactly $n-1$ generically on X. Thus, for a generic $x \in X$, the kernel of the matrix $\mathbf{L}(\mathbf{x})$ is a one dimensional subspace of k^n, and so determines a well defined point in $\mathbb{P}^{n-1}$.

It is easy to check that this map is birational: the genericity hypothesis on the linear forms guarantees that for distinct general elements x, y in X, the matrices $\mathbf{L}(x)$ and $\mathbf{L}(y)$ have distinct null spaces.

As above, the singular locus of X is defined by the vanishing of the $(n-1)\times(n-1)$ subdeterminants of the matrix of linear forms. By the above dimension formula, they define a subset of codimension at most three. Thus a determinental variety in $\mathbb{P}^n$ is never smooth for $n \geq 4$. □

Solution to 1.47.

This is sometimes called Tsen's theorem (Tsen, 1933). The case $d = n = 2$ is due to Max Noether (1870); see Ding *et al.* (1999) for a nice history of the result. We seek solutions $x_i = \sum_{j=0}^{m} a_{ij} t^j$, where the a_{ij} are unknown elements of $\mathbb{C}$, to the degree d polynomial $F(X_0, \ldots X_n)$. Plugging in $X_i = x_i$, and gathering up all terms t^r, we see that the coefficient of t^r is a polynomial in the unknowns a_{ij}. We have a solution if and only if we can choose the a_{ij} so that the coefficient of each t^r is zero. Note that a collection of complex numbers a_{ij} is a solution if and only if λa_{ij} is a solution, where λ is a non-zero scalar, so the solutions naturally live in a projective space over $\mathbb{C}$.

Without loss of generality, we can assume that the coefficients of F are polynomials in t, say of degree at most c. In this case, the highest occurring power of t in the expansion of $F(x_0, \ldots, x_n)$ is at most $dm + c$. Thus to solve for the a_{ij} is to solve a system of $dm + c$ equations in $m(n+1)$ unknowns. Since $n \geq d$, there are more unknowns than equations when $m \gg 0$, so there are solutions for the a_{ij} in projective space over $\mathbb{C}$.

The argument for $\mathbb{C}(t, s)$ is similar. Of course the same argument works with any algebraically closed field in place of $\mathbb{C}$. If $d > n$, the degree d hypersurface in $\mathbb{P}^n$ may have no $\mathbb{C}(t)$ points. For example, there are no $\mathbb{C}(t)$ points of the variety defined by $\sum_{i=0}^{d-1} t^i X_i^d = 0$ in $\mathbb{P}^{d-1}$. By projectivizing the affine cone over this example, we get an example of a degree d hypersurface in $\mathbb{P}^d$ that has exactly one $\mathbb{C}(t)$-point, $(0 : 0 : \cdots : 0 : 1)$. □

Solution to 1.48.

Think of $X_{a,2} \subset \mathbb{A}^1 \times \mathbb{P}^n$ as defined by an equation $\sum_{ij} a_{ij}(t) X_i X_j$ where the $a_{ij} \in \mathbb{C}(t)$ have degree less than or equal to a. This is a quadric in $\mathbb{P}^n_{\mathbb{C}(t)}$ By Exercise 1.47, we know that this quadric has a $\mathbb{C}(t)$-rational point provided $n \geq 2$. This makes the quadric rational over $\mathbb{C}(t)$, whence its function field is isomorphic to $\mathbb{C}(t)(x_1, \ldots, x_n) \cong \mathbb{C}(t, x_1, \ldots, x_n)$. This proves that $X_{a,2}$ is a rational $\mathbb{C}$-variety.

Geometrically, this means that the projection $X_{a,2} \to \mathbb{P}^1$ has a section, making the family birationally trivial over an open set.

The proof of (2) is similar. $\square$

SOLUTION TO 1.49.

Without loss of generality, the point may be assumed to be the affine origin $P = (0, \dots, 0)$. Since X passes through P, it has affine equation

$$f_m(x_1, \dots, x_n) + f_{m+1}(x_1, \dots, x_n) + \cdots + f_d(x_1, \dots, x_n)$$

where each f_i is homogeneous of degree i.

Any line through P has the form

$$(a_1t, a_2t, \dots, a_nt); \quad t \text{ ranging through } k$$

where $(a_1, \dots, a_n)$ is a point on $\mathbb{P}^{n-1}$. To find such a line on X, we need

$$f_m(a_1t, \dots, a_nt) + f_{m+1}(a_1t, \dots, a_nt) + \cdots + f_d(a_1t, \dots, a_nt) = 0$$

for all t. Because each polynomial $f_i(a_1t, \dots, a_nt)$ is homogeneous of degree i in t, we need $(a_1, \dots, a_n)$ such that each $f_i(a_1, \dots, a_n) = 0$. We have $d + 1 - m$ equations on $\mathbb{P}^{n-1}$, so the space of common solutions has dimension at least $n - 1 - (d + 1 - m)$. $\square$

SOLUTION TO 1.50.

If $d < n$, there is even a line through every point by Exercise 1.49. When $d = n$, there may not be a line on X through P. Instead, we prove that through every point on a degree n hypersurface in $\mathbb{P}^n$, there passes a plane conic.

Let $\mathcal{C}$ be the variety of plane conics in $\mathbb{P}^n$ passing through P. The linear system of plane conics forms a projective space of dimension five, and those passing through P is a hyperplane in this space. So the variety of conics through P is a $\mathbb{P}^4$-bundle over the Grassmannian of planes in $\mathbb{P}^n$ through P. This Grassmannian has dimension $2(n - 2)$, so the dimension of the variety $\mathcal{C}$ of conics is $2n$.

The hypersurfaces of degree n in $\mathbb{P}^n$ passing through P naturally form a hyperplane $\mathcal{X}$ in the $\binom{2n}{n} - 1$-dimensional projective space of all degree n hypersurfaces in $\mathbb{P}^n$. Consider the incidence correspondence

$$\Gamma = \{(X, Q) \mid Q \subset X\} \subset \mathcal{X} \times \mathcal{C}$$

together with the two projections $\pi : \Gamma \to \mathcal{X}$ and $q : \Gamma \to \mathcal{C}$.

The elements in the fiber of π over a hypersurface $X \in \mathcal{X}$ can be identified with the conics on X through P. In order to show that through every point on a degree n (or less) hypersurface in $\mathbb{P}^n$ there passes a conic, we need to show that the projection $\Gamma \stackrel{\pi}{\to} \mathcal{X}$ is surjective.

We compute the dimension of Γ using the other projection $q : \Gamma \to \mathcal{C}$. Fix a conic Q through P. We need to compute the dimension of $q^{-1}(Q)$, the space of degree n hypersurfaces containing Q. Choose coordinates so that Q is given by $x_3 = x_4 = \cdots = x_n = 0$ and a homogeneous degree 2 polynomial $g(x_0, x_1, x_2)$. The hypersurfaces of degree m containing Q can be written uniquely in the form:

$$\begin{aligned} &x_n h_n(x_0, \ldots, x_n) \ + \ x_{n-1}h_{n-1}(x_0, \ldots, x_{n-1}) + x_{n-2}h_{n-2}(x_0, \ldots, x_{n-2}) \\ &\quad + \cdots + x_3 h_3(x_0, x_1, x_2, x_3) + g \cdot h(x_0, x_1, x_2), \end{aligned}$$

where the h_i are homogeneous of degree $m - 1$ and h is homogeneous of degree $m - 2$. When $m = n$, the space of hypersurfaces of this form is of dimension $\binom{2n}{n} - 2n - 2$. So the dimension of Γ is

$$\dim q^{-1}(Q) + \dim \mathcal{C} = \left(\binom{2n}{n} - 2n - 2\right) + 2n = \binom{2n}{n} - 2.$$

Now, because Γ and $\mathcal{X}$ have the same dimension, the projection map $\pi : \Gamma \to \mathcal{X}$ is surjective if the fiber over some point in the image is of dimension zero. So the proof is complete upon exhibiting any particular hypersurface of degree n containing only finitely many conics through a point P. We leave it to the reader to verify that the hypersurface defined by $X_0^n - X_1X_2\ldots X_n$ contains only finitely many conics through the point $(1 : 1 : \cdots : 1 : 1)$. $\square$

Solution to 1.51.

Fix a general hyperplane L in $\mathbb{P}^{3n+1}$. Pick a point p in L and project the three linear spaces from p to some other general hyperplane. After projection, the linear spaces intersect in a single point, so there is a unique line through p which intersects all three linear spaces. This line intersects X in a unique fourth point. Sending p to this unique fourth point gives a birational map from L to X. $\square$

Solution to 1.53.

Use the exact sequence

$$0 \to \Omega_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(n+1)} \to \mathcal{O}_{\mathbb{P}^n} \to 0$$

(see Hartshorne, 1997, p. 176 for the derivation of this sequence). Since $\Omega_{\mathbb{P}^n}$ is a subsheaf of $\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(n+1)}$, it follows that

$$(\Omega_{\mathbb{P}^n})^{\otimes m} \subset (\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(n+1)})^{\otimes m} \cong \mathcal{O}_{\mathbb{P}^n}(-m)^{\oplus(n+1)^m}.$$

But then $\Omega_{\mathbb{P}^n}^{\otimes m}$ has no nonzero global sections, since $\mathcal{O}_{\mathbb{P}^n}(-m)$ has none. $\square$

Solution to 1.59.

By the adjunction formula, the canonical class of a hypersurface X in $\mathbb{P}^n$ is $K_X = (K_{\mathbb{P}^n} + X)|_X$, so $\mathcal{O}_X(K_X) = \mathcal{O}_X((-n-1+d)H)$ where X is degree d. So for $d > n$, all powers of $\mathcal{O}_X(K_X)$ have nonzero global sections and the plurigenera do not vanish. $\square$

Solution to 1.60.

A variety X/k of dimension d is unirational if and only if its function field has an algebraic extension that is purely transcendental over k. If $X' \to X$ is purely inseparable, then by definition, the function fields have the following relationship

$$\{k(X')\}^{p^e} \subset k(X) \subset k(X')$$

where p^e is some power of the characteristic p. If X is unirational, then $k(X)$ has an algebraic extension $k(t_1, \dots, t_d)$, and hence $k(X') \subset k^{1/p^e}(t_1^{1/p^e}, \dots, t_d^{1/p^e})$. Since k is perfect, $k = k^{1/p^e}$, and $k(X')$ is a subfield of a purely transcendental extension of k. Thus X' is unirational over k. $\square$

7.2 Exercises in Chapter 2

Solution to 2.6.

Note that $E^2 = -1$, and that, since P has multiplicity m on C, $C' \cdot E = m$. Thus $C^2 = (C' + mE) \cdot (C' + mE) = (C')^2 + 2mC' \cdot E + m^2E^2 = (C')^2 + m^2$. For the other equality, first verify that $K_{S'} \cdot E = -1$ using the adjunction formula $\deg K_E = (K_{S'} + E) \cdot E$. Then compute that $C \cdot K_S = (C' + mE) \cdot (K_{S'} - E) = C' \cdot K_{S'} - C' \cdot E + mE \cdot K_{S'} - mE^2 = C' \cdot K_{S'} - m$. $\square$

Solution to 2.8.

Let $\pi_i : S_i \to S_{i-1}$ denote the ith blowup in our sequence resolving the indeterminacies, say of the base point P_i whose multiplicity is m_i. Let Γ_i denote the birational transform of Γ on S_i. (So $S = S_0$ and $\Gamma = \Gamma_0$.) From Exercise 2.6, we have that

$$\Gamma_i^2 = \Gamma_{i-1}^2 - m_i^2 \quad \text{and} \quad \Gamma_i \cdot K_{S_i} = \Gamma_{i-1} \cdot K_{S_{i-1}} + m_i,$$

for each i. Concatenating the formulas, we get

$$\bar{\Gamma}^2 = \Gamma^2 - \sum m_i^2 \quad \text{and} \quad \bar{\Gamma} \cdot K_{\bar{S}} = \Gamma \cdot K_S + \sum m_i.$$

Now since the linear system $\bar{\Gamma}$ is the pull-back of the hyperplane system on T and $K_{\bar{S}}$ agrees with K_T except along exceptional divisors, we see that

$$\bar{\Gamma} \cdot K_{\bar{S}} = \phi_{\bar{\Gamma}}^* H \cdot (\phi_{\bar{\Gamma}}^* K_S + \phi_{\bar{\Gamma}}\text{-exceptional divisors}) = H \cdot K_T.$$

Finally, since $\phi_{\bar{\Gamma}}$ is birational, formula (2.5.1) implies that $\bar{\Gamma}^2 = \deg(T)$. The proof is complete. $\square$

SOLUTION TO 2.15.

The map $q : S' \to \mathbb{P}^2 = \mathbb{P}(T_P\mathbb{P}^3)$ can be described as follows: for $Q \in S'$ but not in the exceptional fiber, thinking of Q as a point in S, $q(Q)$ is the line L through P and Q; for Q in the exceptional fiber, thinking of Q as a direction at P, $q(Q)$ is the line through P in the direction of Q.

Clearly the fiber of q over a point $L \in \mathbb{P}^2$ consists of the two points Q_1 and Q_2 that, together with P, make up the intersection $L \cap S$. Ramification occurs precisely when $Q_1 = Q_2$. To find the equation of this ramification locus, choose coordinates so that $P = (0 : 0 : 0 : 1)$ is the origin in an affine chart where the surface S is given by an equation of the form

$$f_1(x, y, z) + f_2(x, y, z) + f_3(x, y, z)$$

with f_i homogeneous of degree i.

A line L through P is given by parametric equations (at, bt, ct) corresponding to a point $(a : b : c)$ in $\mathbb{P}^2$. The intersection points of this line with S are given by the solutions of the equation

$$tf_1(a, b, c) + t^2 f_2(a, b, c) + t^3 f_3(a, b, c) = 0.$$

The two solutions (other than $t = 0$) define the fiber over L under the map q. Ramification occurs when the two solutions are identical, so it is given by the discriminant. Thus the ramification locus in $\mathbb{P}^2$ is the quartic defined by the homogeneous equation $f_2^2 - 4 f_1 f_3$.

To see that this ramification locus is smooth, note that because S' is a degree two cover of $\mathbb{P}^2$, locally S' is defined by a quadratic polynomial of the form $u^2 - g(s, t)$, where s, t are local coordinates on $\mathbb{P}^2$. The ramification locus is locally defined by $g = 0$. On the other hand, since the polynomial $u^2 - g(s, t)$ defines a smooth variety (namely S'), the Jacobian criterion implies that $g(s, t)$ also defines a smooth variety in $\mathbb{P}^2$. $\square$

SOLUTION TO 2.17.

In order to prove (1), represent S as $\mathbb{P}^2$ blown up in six points $P_1, \ldots, P_6$. Thus the Picard group of S is generated by the pull-back f^*H of a general line $H \subset \mathbb{P}^2$ and the six exceptional curves $E_1, \ldots, E_6$. The pull-back f^*H can be written as the birational transform E_0 of a line through P_1, P_2 (which is a line on the cubic) plus $E_1 + E_2$.

Assume that D_1, D_2 are two divisors on S such that $L \cdot D_1 = L \cdot D_2$ for every line. By (1) we can write

$$D_1 - D_2 \sim a_0 f^* H + \sum_{i=1}^{6} a_i E_i .$$

We prove that $a_i = 0$ using that $L \cdot (D_1 - D_2) = 0$ for every line. For $i > 0$ this follows from $a_i = -E_i \cdot (D_1 - D_2) = 0$. For $i = 0$ we use $a_0 = f^* H \cdot (D_1 - D_2) = (E_0 + E_1 + E_2) \cdot (D_1 - D_2) = 0$.

Let us prove finally (3). We use only that S has a hyperplane section which is a sum of three lines. (In the six point blowup these are given, for instance, by three lines in the plane that pass through all six points. Usually, however, such hyperplane sections are found on the way to proving the rationality of cubics, cf. Shafarevich (1994, IV.2.5).

The proof is by induction on the intersection number $D \cdot H$ where $H = -K_S$ is the hyperplane class. It is enough to consider the case when D is irreducible and reduced. If $D \cdot H = 1$ then D is a line. If $D^2 < 0$ then D is a line. Indeed, from the adjunction formula $-2 \leq 2g(D) - 2 = D^2 + D \cdot K$. On a cubic $D \cdot K < 0$ for any curve, so if $D^2 < 0$ then the only possibility is $D^2 = D \cdot K = -1$, hence D is a line.

By Riemann–Roch,

$$h^0(S, \mathcal{O}_S(D)) - h^1(S, \mathcal{O}_S(D)) + h^2(S, \mathcal{O}_S(D)) = \frac{D \cdot (D - K_S)}{2} + 1.$$

By Serre duality, $h^2(S, \mathcal{O}_S(D)) = h^0(S, \mathcal{O}_S(K_S - D)) = 0$. If D^2 is negative, then D is a line; otherwise we get that

$$h^0(S, \mathcal{O}_S(D)) \geq \frac{D \cdot (D - K_S)}{2} + 1 \geq \frac{D \cdot (-K_S)}{2} + 1.$$

Let now $L \subset S$ be any line and consider the sequence

$$0 \to \mathcal{O}_S(D - L) \to \mathcal{O}_S(D) \to \mathcal{O}_L(D|_L) \to 0.$$

We conclude that $D - L$ is linearly equivalent to an effective divisor if

$$h^0(S, \mathcal{O}_S(D)) - h^0(S, \mathcal{O}_L(D|_L)) = h^0(S, \mathcal{O}_S(D)) - D \cdot L - 1 > 0.$$

By the above estimate, this holds if $-D \cdot (-K_S) > 2D \cdot L$. Since $-K_S$ is a hyperplane section, it is linearly equivalent to a sum of three lines $L_1 + L_2 + L_3$. Thus $D \cdot L_i \leq D \cdot (-K_S)/3$ for some i. Choosing $L = L_i$ we obtain that $D - L$ is linearly equivalent to an effective divisor and the inductive step is finished. $\square$

SOLUTION TO 2.18.

Consider a line L in the ambient three-space, together with the smooth cubic surface S defined by $u^3 = f(x, y)$. We claim that if the line L lies on S, then L projects to a line triply tangent to the smooth plane cubic curve $(f(x, y) = 0)$ in the xy-plane. Indeed, let L' denote this projection, and suppose it has equation $y = mx + b$. The line L' is triply tangent if and only if $f(x, mx + b)$ is a cube of a linear form, say $(cx + d)^3$. Whenever we have this perfect cube, there are three lines on S projecting to L'. These three lines have parametric equations

$$(x,\ mx + b,\ \omega(cx + d))$$

where ω is one of the three cube roots of unity.

Because the plane cubic $\{f(x, y) = 0\}$ has nine points of triple tangency, all twenty-seven lines in S are constructed in this way.

We work out explicitly the lines on the Fermat surface given in homogeneous coordinates by

$$a_0X_0^3 + a_1X_1^3 + a_2X_2^3 + a_3X_3^3.$$

Factoring the first two two terms $a_0X_0^3 + a_1X_1^3$ completely into distinct homogeneous linear polynomials $l_1l_2l_3$, and likewise factoring $a_2X_2^3 + a_3X_3^3 = m_1m_2m_3$, the Fermat cubic has the form

$$l_1l_2l_3 + m_1m_2m_3 = 0.$$

The linear factors are distinct because the surface is smooth. This produces nine lines on the surface, defined by the nine different pairs of planes containing them:

$$\{l_i = m_j = 0\}_{ij}.$$

By considering factorizations of the other two groupings of the terms $(a_0X_0^3 + a_2X_2^3) + (a_1X_1^3 + a_3X_3^3)$ and $(a_0X_0^3 + a_3X_3^3) + (a_1X_1^3 + a_2X_2^3)$ we can produce the remaining eighteen lines on the surface in similar fashion.

The case where $u^2 = f(x, y)$ is similar. However, there are three distinct lines on the surface in the plane at infinity (meeting in an Eckardt point). The remaining 24 lines on S project to twelve lines in the plane $(u = 0)$ which pass through one of the points at infinity of the smooth curve defined by $(f = 0)$ and are tangent to it at another point. These twelve lines can be found by solving for m and b such that $f(x, mx + b)$ is a perfect square.

We show that the cubic surface defined by $x_1^3 + x_2^3 + x_3^3 = a$ (where a is not a cube) is not rational over $\mathbb{Q}$. By Segre's theorem, it suffices to show that the

Picard number is one, and by (2.16), it is enough to show that no Galois orbit consists of disjoint lines on the surface.

The computation above indicates that all lines are defined over the splitting field of $t^3 - a$ over $\mathbb{Q}$. This splitting field is degree six over $\mathbb{Q}$, and is generated by ω, a primitive third root of unity and a real third root β of a. The Galois group is the full group S_3 of all permutations of the roots β, $\beta\omega$, and $\beta\omega^2$ of $t^3 - a$.

Factoring the equation for the cubic $(x_1^3 + x_2^3) + (x_3^3 + ax_0^3)$ as

$$(x_1 + x_2)(x_1 + \omega x_2)(x_1 + \omega^2 x_2) + (x_3 + \beta x_0)(x_3 + \beta\omega x_0)(x_3 + \beta\omega^2 x_0),$$

we consider the lines as described above.

Any line defined by

$$\{x_1 + x_2 = 0;\ \ x_3 + \beta\omega^i x_0 = 0\}$$

for some $i = 0, 1, 2$ contains all other lines of this type in its orbit, since the Galois group acts transitively on the $\beta\omega^i$. This orbit consists of three lines in the plane $\{x_1 + x_2 = 0\}$; in particular, the lines of this orbit are not disjoint.

Now consider the orbit of a line defined by

$$\{x_1 + \omega x_2 = 0;\ \ x_3 + \beta\omega^i x_0 = 0\}$$

for some $i = 0, 1, 2$. The cyclic permutation $\beta \mapsto \beta\omega \mapsto \beta\omega^2 \mapsto \beta$ fixes ω. So the orbit of this line contains three lines in the plane $\{x_1 + \omega x_2 = 0\}$, and hence can not consist of disjoint lines. Furthermore, the permutation interchanging $\beta\omega$ and $\beta\omega^2$ sends these lines to lines of the form

$$\{x_1 + \omega^2 x_2 = 0;\ \ x_3 + \beta\omega^i x_0 = 0\}.$$

The same cyclic permutation now takes this line to all others of this form, that is, to all others in the plane $\{x_1 + \omega^2 x_2 = 0\}$. So the six lines in these two planes constitute another orbit.

Considering the other two groupings of lines, we find that there are two more orbits consisting of three lines in the same plane, and two more orbits consisting of six lines in two planes. None of these six orbits consists of disjoint lines, so we conclude that the Picard number of the surface is one. □

SOLUTION TO 2.19.

Let k^∞ be the perfect closure $\cup_e k^{1/p^e}$ of k. Then the automorphism groups $\operatorname{Aut}\bar{k}/k$ and $\operatorname{Aut}\bar{k}/k^\infty$ are identical, because k^∞ is the precisely the fixed field of $G = \operatorname{Aut}\bar{k}/k$. The cubic surface S is smooth whether regarded over k or k^∞, and the orbits of the action of G on the twenty-seven lines are independent of this choice. Furthermore, any divisor on S defined over k is *a priori* defined over k^∞, while any divisor defined over k^∞ is defined over some purely inseparable extension k^{1/p^e} of k, so that some p^eth multiple is defined over k. This implies that the Néron–Severi group has the same rank over k or k^∞. In particular, conditions (1), (2), and (3) are equivalent over k if and only if they are equivalent over k^∞. □

Solution to 2.29.

A quadratic transformation is given by a two-dimensional linear system Γ contained in the complete linear system $|2H|$, where H is the hyperplane system on $\mathbb{P}^2$. The general member of $C \in \Gamma$ is irreducible, hence a smooth conic. Remembering that five points determine a conic, we see that our two-dimensional linear system of conics has three base points. Up to coordinate change, then, there are only three possibilities for Γ, depending on whether there are three, two or one distinct geometric base points. These correspond to the three enumerated cases. □

Solution to 2.34.

(1) Let $T : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a de Jonquières map. If $P \in \mathbb{P}^2$ is the base point of multiplicity $n-1$, then after composing with a linear change of coordinates, we may assume that T is an isomorphism on the generic fiber of the ruling of the blown-up surface $\mathbf{F}_1 = \mathrm{Bl}_P\,\mathbb{P}^2$; see Definition 2.28 (1). Choose affine coordinates x, y such that P is the point at infinity on the y-axis. Then the fibers of the ruling are the vertical lines, and T is an isomorphism of $\mathbb{P}^1_{k(x)}$ with affine coordinate y, defined over the ground field $k(x)$. Therefore it can be written as a Möbius fractional linear transformation:

$$(x, y) \to \left(x, \frac{a(x)y + b(x)}{c(x)y + d(x)}\right)$$

where $a, b, c, d \in k(x)$ are rational functions of x. For example the quadratic maps of Exercise 2.29 are written as:

$$T_1 \colon y \to \frac{x}{y}, \quad T_2 \colon y \to xy, \quad T_3 \colon y \to y + x^2.$$

(2) We show that every de Jonquières map is the composition of quadratic (and linear) maps. It is well known and easy to show that the group of fractional linear transformations is generated by three kinds of maps

$$S_1 \colon y \to a(x)y, \quad S_2 \colon y \to \tfrac{1}{y}, \quad S_3 \colon y \to y + b(x).$$

Composing maps of type T_2 and translations in x, we get all $y \to P(x)y$ where $P(x) \in k[x]$ is an arbitrary polynomial in x. Note that $T_1 \circ T_2(y) = 1/y$ and then $T_1 \circ T_2 \circ T_1(y) = y/x$. Hence we get all maps of the form S_1. To generate everything we only need to produce maps S_3. The key observation in this respect is that

$$y + b(x) = b(x)\left(\frac{1}{b(x)}y + 1\right)$$

so we get these as compositions of maps of type S_1 and linear translations in y.

(3) Finally we show that all quadratic maps are the compositions of standard quadratic maps and linear maps. In (2), we have already implicitly shown that a map of type T_3 is a composition of maps of type T_1 (composing with a linear map, these are standard quadratic) and T_2 – indeed, we did not use T_3 in the proof.

The map T_2 is given by the linear system of divisors whose equations are from the vector space spanned by x_0^2, x_0x_1, x_1x_2. This is the linear system of conics passing through the point $P = (0:1:0)$ and through the point $Q = (0:0:1)$ with tangent direction $L_Q = \{x_1 = 0\}$. The idea – see the end of the proof of Theorem 2.32 – is to "untwist" T_2 by a standard quadratic map with two base points at P, Q and a third base point R somewhere else. We only need to make sure that R is not on the line L_Q; in particular R can not be the point $(1:0:0)$. To implement this, let us change coordinates slightly, so T_2 takes the following form

$$T_2'\colon (x_0 : x_1 : x_2) \dashrightarrow \left(x_0^2 : x_0x_1 : (x_0 + x_1)x_2\right).$$

Indeed, this map is given by the linear system of conics passing through the point $P = (0:1:0)$ and through the point $Q = (0:0:1)$ with tangent direction $L_Q' = \{x_0 + x_1 = 0\}$. Untwisting by the standard quadratic map $S\colon (x_0 : x_1 : x_2) \dashrightarrow (x_1x_2 : x_0x_2 : x_0x_1)$ we obtain

$$\begin{aligned} T_2' \circ S\colon (x_0 : x_1 : x_2) &\dashrightarrow \left(x_1^2x_2^2 : x_0x_1x_2^2 : (x_1 + x_0)x_0x_1x_2\right) \\ &= (x_1x_2 : x_0x_2 : (x_1 + x_0)x_0). \end{aligned}$$

This is, up to linear changes of coordinates, a standard quadratic map. Indeed the base locus consists of three actual points, namely $(0:1:0)$, $(0:0:1)$, and $(-1:1:0)$. □

7.3 Exercises in Chapter 3

Solution to 3.4 and 3.40.

(a) We first assume that C is a reduced curve of arithmetic genus zero on a smooth surface S. To prove (1), let $C_1 \subset C$ be an irreducible component. There is an exact sequence

$$0 \to I \to \mathcal{O}_C \to \mathcal{O}_{C_1} \to 0.$$

The corresponding long exact sequence of cohomology shows that that $H^1(\mathcal{O}_{C_1}) = 0$. This forces C_1 to be isomorphic to $\mathbb{P}^1$.

Next we prove parts (2) and (3) and Exercise 3.40 together. Let S be a smooth surface and $C \subset S$ a reduced and connected curve. Pick a point $P \in C$ of multiplicity m and blow it up to get $\pi : S' \to S$. Let $C' := \pi_*^{-1}C$ be the birational transform of C on S'. Then $\pi^* C = C' + mE$ where $E \subset S'$ is the exceptional curve. Thus

$$\begin{aligned} -2\chi(\mathcal{O}_{C'}) &= C' \cdot (C' + K_{S'}) \\ &= (\pi^* C - mE) \cdot (\pi^* C - mE + \pi^* K_S + E) \\ &= C \cdot (C + K_S) - m(m-1) = -2\chi(\mathcal{O}_C) - m(m-1). \end{aligned}$$

Because C' has at most m connected components, it follows that $\chi(\mathcal{O}_{C'}) \le m$. So

$$h^1(C, \mathcal{O}_C) = 1 - \chi(\mathcal{O}_C) \ge \frac{m(m-3)}{2} + 1,$$

which completes the proof of Exercise 3.40. Assume now that $H^1(C, \mathcal{O}_C) = 0$. We get that m is at most two, and if $m = 2$ then

$$0 = h^1(C, \mathcal{O}_C) = 2 - \#\{\text{connected components of } C'\}.$$

Thus blowing up any singular point disconnects C. This means that any singular point of C is a transverse intersection of components, and that the components form a tree. This completes the proof of (2) and (3).

To prove the converse statement, it turns out to be easiest to first prove (b). Let us assume that a curve C satisfies statements (1), (2), and (3). We will show that it satisfies the conclusion of (b), and then the converse part of (a).

In order to prove (b), we may assume that $C = \sum C_i$ is connected. By (3), there is an irreducible component $C_1 \subset C$ such that $C_1 \cdot C^* = 1$, where $C = C_1 + C^*$. There is an exact sequence

$$0 \to L|_{C_1}(-1) \to L \to L|_{C^*} \to 0$$

This gives (b) by induction on the number of irreducible components.

Now to complete the proof of (a), note that we used only the properties (1), (2), and (3) in proving (b). Thus given a curve satisfying (1), (2), and (3), we can apply (b) to the $L = \mathcal{O}_C$ case. This shows that $h^1(C, \mathcal{O}_C) = 0$, and the proof is complete.

(c). Let k be a perfect field and $D \subset S$ a proper reduced irreducible curve such that $D_{\bar{k}}$ is connected. Then the irreducible components of $D_{\bar{k}}$ form a single Galois orbit. The Galois group acts on the tree $G(D_{\bar{k}})$ and the ends of the branches form one orbit. Thus everything is an end and this happens only if there are at most two curves.

In the smooth case we have a plane conic by Proposition 1.4. A similar argument works in the reducible case. The embedding given by sections of ω_D^{-1} realizes D as a reducible conic over $\bar{k}$, so the same holds over k by Exercise 3.34. □

SOLUTION TO 3.8.

The ideal of P is $(x, y^p - v)$. Thus the blow up is given in $\mathbb{A}^2_{x,y} \times \mathbb{P}^1_{s,t}$ by a single equation $xs - (y^p - v)t = 0$. Over the algebraic closure we can introduce a new coordinate $y_1 = y - \sqrt[p]{v}$ and there the equation becomes $xs - y_1^p t = 0$. The resulting hypersurface is not smooth at $x = s = y_1 = 0$.

The ideal of Q is $(x^p - u, y^p - v)$ and the blowup is given by the equation $(x^p - u)s - (y^p - v)t = 0$. Over the algebraic closure we can introduce new coordinates $x_1 = x - \sqrt[p]{u}$, $y_1 = y - \sqrt[p]{v}$ and there the equation becomes $x_1^p s - y_1^p t = 0$. The corresponding hypersurface is not smooth along the line $x_1 = y_1 = 0$.

We still need to prove that these hypersurfaces are regular. This is easiest done using the following principle:

LEMMA 7.1. *Let $f : X \to Y$ be a morphism of k-varieties, X smooth. Then the generic fiber X_{gen} of f is regular as a variety over $k(Y)$.*

PROOF. This holds since every local ring of X_{gen} is also a local ring of X and every local ring of a smooth scheme is regular (cf. Matsumura (1980, 18.G) or Matsumura (1986, 19.3)). □

The scheme $B_P\mathbb{A}^2$ can be viewed as the generic fiber of the $k(u)$-morphism

$$X := (xs - (y^p - v)t = 0) \subset \mathbb{A}^3_{x,y,v} \times \mathbb{P}^1_{s,t} \xrightarrow{\pi} \mathbb{A}^1_v =: Y$$

where π is a coordinate projection. It is easy to check that X is smooth.

Similarly, $B_Q\mathbb{A}^2$ is the generic fiber of the k-morphism

$$X := ((x^p - u)s - (y^p - v)t = 0) \subset \mathbb{A}^3_{x,y,u,v} \times \mathbb{P}^1_{s,t} \xrightarrow{\pi} \mathbb{A}^2_{u,v} =: Y.$$

□

SOLUTION TO 3.9.

Over $\bar{k}$ the curve C decomposes as $C_{\bar{k}} = C_1 + \cdots + C_n$. Then

$$C \cdot K_S = nC_1 \cdot K_S \quad \text{and} \quad C^2 = nC_1^2 + nC_1 \cdot (C_2 + \cdots + C_n).$$

Thus if $C \cdot K_S < 0$ and $C^2 < 0$ then $C_1 \cdot K_S < 0$ and $C_1^2 < 0$, hence C_1 is a -1-curve. We also obtain that

$$C_1 \cdot (C_2 + \cdots + C_n) = -C_1^2 + \tfrac{1}{n} C^2 < 1.$$

Thus C_1 is disjoint from the other curves C_i and so $C_1, \ldots, C_n$ is a Galois orbit of disjoint -1-curves over $\bar{k}$. □

SOLUTION TO 3.13.

(1) Let $\mathbb{P}^1 \cong F \subset S$ be a smooth fiber. Then $F^2 = 0$ and

$$-2 = F \cdot (F + K_S) = F \cdot K_S.$$

If $2F'$ is a double line fiber, then $F' \cdot K_S = \frac{1}{2} F \cdot K_S = -1$. Thus

$$2g(F') - 2 = F' \cdot (F' + K_S) = -1,$$

a contradiction, showing that there are no double line fibers. Finally, let $F_1 + F_2$ be a reducible fiber. Then

$$F_i^2 = -F_i \cdot F_{3-i} = -1,$$

and using the adjunction formula, we verify that each F_i is a -1-curve.

(2) Note that there are at most finitely many reducible fibers. In each reducible fiber we can contract one of the irreducible components, to get a surface $T \to C$ where every fiber is irreducible. If $G \subset T$ is any fiber, then $G \cdot (G + K_T) = G \cdot K_T = -2$ since this holds for a general fiber. Thus $G \cong \mathbb{P}^1$ and $T \to C$ is a $\mathbb{P}^1$-bundle.

(3) With T as constructed in (2), note that $K_T^2 = 8(1 - g(C))$ by Hartshorne (1977, V.2.11). Since each blowup decreases K^2 by 1, K_S^2 can not be larger than K_T^2, and (3) is proved.

(4) The above computations also give that $\mathcal{O}(-K_S)$ restricted to any fiber is the same as $\mathcal{O}_{\mathbb{P}^2}(1)$ restricted to the corresponding plane conic. Thus $h^0(F, \mathcal{O}(-K_S)|_F) = 3$ and $h^1(F, \mathcal{O}(-K_S)|_F) = 0$ for any fiber. Thus $f_*\mathcal{O}(-K_S)$ is a rank three vector bundle by Lemma 7.2 below. The natural embedding $S \hookrightarrow \mathbb{P}_C f_*\mathcal{O}(-K_S)$ also follows.

(5) Let $g : T \to C$ be a morphism with connected fibers such that $-K_T$ is g-ample, and suppose that $F \subset T$ is an irreducible fiber of g. Then $F \cdot (F + K_T) = F \cdot K_T < 0$, hence $F \cong \mathbb{P}^1$ and $F \cdot K_T = -2$. Let now $\sum m_i F_i$ be any fiber. Then $-2 = (\sum m_i F_i) \cdot K_T \leq -\sum m_i$, hence either we have two components or one component which can be simple or double. Computing the genus of F_i as above gives that $T \to C$ is a conic bundle.

(6) Let T be a smooth surface over a field k and $f : T \to C$ a morphism whose general fiber is a smooth rational curve. Over $\bar{k}$, let $\sum m_i F_i$ be a fiber

with $\sum m_i \geq 2$. Then $(\sum m_i F_i) \cdot K_T = -2$, so there is an index, say $i = 1$, such that $F_1 \cdot K_T < 0$. Write $\sum m_i F_i = m_1 F_1 + F'$. If $F' \neq 0$ then $F_1^2 = -\frac{1}{m_1} F_1 \cdot F' < 0$, hence F_1 is a -1-curve. The conjugates of F_1 are also -1-curves in fibers of f, thus we obtain a -1-curve over k unless there are two intersecting conjugates in the same fiber.

Thus assume that F_1, F_2 is a pair of intersecting -1-curves in a fiber and assume that $m_1 \leq m_2$. $F_1 \cdot (m_1 F_1 + F') = 0$ implies that $F_1 \cdot F' = m_1$. But F' contains F_2 with multiplicity m_2. Thus $m_1 = m_2$, $F_1 \cdot F_2 = 1$ and there are no other components. This means that all the conjugates of F_1 sit in fibers where $T \to C$ is a conic bundle. Hence if there are no -1-curves at all then $T \to C$ is a conic bundle.

The following is a special case of Grauert's theorem (cf. Hartshorne, 1977 III.12.9).

LEMMA 7.2. *Let $f : X \to C$ be a morphism from a projective variety to a smooth curve C. Let $X_c := f^{-1}(c)$ denote the fiber over $c \in C$. Let M be a line bundle on X such that $h^1(X_c, M|_{X_c}) = 0$ for every $c \in C$. Then there is an n such that if L is a line bundle on C with* $\deg L \geq n$ *and $c \in C$ is any point then the restriction*

$$H^0(X, M \otimes f^*L) \to H^0(X_c, (M \otimes f^*L)|_{X_c}) \quad \textit{is surjective.}$$

PROOF. For $c \in C$ we get an exact sequence

$$0 \to M \otimes f^*L \to M \otimes f^*L(c) \to (M \otimes f^*L(c))|_{X_c} \to 0,$$

whose long exact cohomology sequence gives a surjection

$$H^1(X, M \otimes f^*L) \twoheadrightarrow H^1(X, M \otimes f^*L(c)).$$

From this we find an n such that if $\deg L \geq n - 1$ then $h^1(X, M \otimes f^*L)$ is independent of L. Now we look at

$$0 \to M \otimes f^*L(-c) \to M \otimes f^*L \to (M \otimes f^*L)|_{X_c} \to 0.$$

We get a surjection

$$H^1(X, M \otimes f^*L(-c)) \twoheadrightarrow H^1(X, M \otimes f^*L)$$

of vector spaces of the same dimension. This is therefore an isomorphism and so the map

$$H^0(X, M \otimes f^*L) \to H^0(X_c, (M \otimes f^*L)|_{X_c})$$

is surjective. $\square$

The proof of Exercise 3.13 is complete. $\square$

Solution to 3.19.

Let B_i be the irreducible components of B, and assume that $B \cdot B_i \geq 0$ for every B_i. Let $B = B' + B''$ be any decomposition into two nonintersecting parts. Then B'^2 and B''^2 are both non-negative.

If B' is numerically equivalent to a multiple of B'' then $B' \cdot B_i$ is a multiple of $B'' \cdot B_i$. If B_i is irreducible, it is disjoint from one of B', B'', hence $B' \cdot B_i$ and $B'' \cdot B_i$ are both zero.

Otherwise, B'^2, B''^2 can not both be non-negative since by the Hodge index theorem (see Hartshorne, 1977, V.1.9.1) the intersection form on divisors has only one positive eigenvalue. $\square$

Solution to 3.20.

Let $X \subset \mathbb{P}^N$ be a scheme and $H \subset \mathbb{P}^N$ a general hyperplane. If $H \cap X$ is irreducible, then so is X. By repeatedly applying this, we reduce Exercise 3.20 to the case when M is a linear system on a surface X.

M gives a rational map $X \dashrightarrow Z \subset \mathbb{P}^n$. The linear system M is composed with a pencil if and only if $\dim Z = 1$. Thus assume that $\dim Z \geq 2$. Let Y be the normalization of Z in the function field of X. Our map factors as $X \dashrightarrow Y \to Z$ where $Y \to Z$ is a finite morphism given by a base point free linear system N.

Since $X \dashrightarrow Y$ is birational, it is sufficient to prove that the general member of N is irreducible. As Y is a normal surface, it has only isolated singularities. In characteristic zero we can use the usual Bertini theorem as in Hartshorne (1977, III.10.9) to conclude that a general member of N is smooth. Every member of N is connected by Hartshorne (1977, III.7.9), hence we obtain that a general member of N is irreducible.

In positive characteristic smoothness no longer holds and we have to proceed more carefully.

Let P be the projective space of the elements of N and $P^0 \subset P$ those divisors that do not pass through any singular point of Y. Let $I \subset Y \times P$ be the set of pairs (y, N) satisfying $y \in N$. The map $I \to Y$ is a hyperplane bundle, thus I is smooth away from the preimages of the singular points of S. This means that there is a smooth open subset $I^0 \subset I$ such that $I^0 \to P^0$ is proper. The fibers of $I \to P$ are connected by Hartshorne (1977, III.7.9).

By Lemma 7.1 in the proof of Exercise 3.8, we obtain that the generic fiber of $I \to P$ is regular and connected. A regular and connected scheme is geometrically irreducible, but this is not easy to prove. Here we present a somewhat modified proof using the smoothness of I^0. A similar argument settles the general case once we know that a ring smooth over a regular ring is regular.

In our case we are reduced to proving the following.

LEMMA 7.3. *Let $g : U \to V$ be a proper morphism with connected fibers, U smooth. Then the geometric generic fiber of g is irreducible.*

PROOF. Let U^* denote the generic fiber as a $k(V)$ variety. We need to prove that U^*_L is irreducible for every finite extension $L \supset k(V)$. We may assume that $L/k(V)$ is normal, hence there is a subextension $L \supset K \supset k(V)$ such that L/K is purely inseparable and $K/k(V)$ is separable.

Let V' be the normalization of V in K. Then $V' \to V$ is smooth over a dense set, hence $U \times_V V'$ is smooth along the generic fiber. As we noted in Lemma 7.1 this implies that U^*_K is regular. A connected and regular scheme is irreducible.

The map $U^*_L \to U^*_K$ is purely inseparable, hence a homeomorphism. Therefore U^*_L is also irreducible. □

The proof of the Bertini theorem is complete. □

SOLUTION TO 3.23.

The easiest examples are found when K has very few algebraic extensions, for instance when $K = \mathbb{R}$. Let S be any conic bundle without -1-curves which is not a $\mathbb{P}^1$-bundle. Over $\mathbb{C}$, each singular fiber gives us two -1-curves. To get explicit examples, take, for instance, the conic bundle $S \to \mathbb{P}^1$ where S is the hypersurface in $\mathbb{P}^1 \times \mathbb{P}^2$ defined by the vanishing of

$$f_x(s,t)x^2 + f_y(s,t)y^2 + f_z(s,t)z^2$$

where s and t are homogeneous coordinates for $\mathbb{P}^1$ and x, y and z are homogeneous coordinates for $\mathbb{P}^2$. If every root of $f_x f_y f_z = 0$ is real then the -1-curves appear over real points. If, in addition, at every root the other two factors have the same sign, then all -1-curves appear in conjugate intersecting pairs. For instance one can take

$$f_x = s^2 - a^2t^2,\ f_y = (s-t)^2 - a^2t^2,\ f_z = (s+t)^2 - a^2t^2 \quad \text{for } 0 < a < \tfrac{1}{2}.$$

□

SOLUTION TO 3.27.

If L is defined over C_K then the unique zero of any section of L gives a K-point of C. There are two ways to see that C has no $\mathbb{Q}$-points. One can use that 3 is not a square modulo 5 or that 5 is not a square modulo 3. □

SOLUTION TO 3.31.

Let L be the minimal field of definition of W. If $g_1(W) = g_2(W)$ then $W = g_1^{-1}g_2(W)$. Thus W is also defined by equations in $g_1g_2^{-1}(L)$ (this is how the dual action on functions works out) and so $L = g_1g_2^{-1}(L)$, or

$g_1^{-1}(L) = g_2^{-1}(L)$. Thus $\{g(W) : g \in \mathrm{Aut}(K/k)\}$ is in one-to-one correspondence with the images of L under k-automorphisms of K. It is a standard result of algebra that this set is finite if and only if L/k has finite degree. (The key point is to prove that every embedding $L \to K$ extends to the algebraic closure $\bar{L} \to K$.) □

SOLUTION TO 3.32.

Let $\bar{K}$ be an algebraic closure of K. For every $g \in \mathrm{Aut}(\bar{K}/k)$ the curve $g(C)$ has the same degree and self-intersection as C.

Basic facts about the Hilbert scheme imply that the set of curves $\{g(C) : g \in \mathrm{Aut}(\bar{K}/k)\}$ is finite. (For our purposes the results explained in Shafarevich (1994, VI.4) are enough.) Hence C is defined over an algebraic extension of k by Exercise 3.31. On the other hand, C is defined over K which has no nontrivial algebraic subextensions. Thus C is defined over k by Theorem 3.26. □

SOLUTION TO 3.34.

Let U_i be an affine cover of X. The $H^i(X, F)$ are computed as cohomologies of the Čech-complex

$$0 \to \sum_i \Gamma(U_i, F|_{U_i}) \to \sum_{i<j} \Gamma(U_i \cap U_j, F|_{U_i \cap U_j}) \to \cdots .$$

If $F|_{U_i}$ is the sheaf associated to the $k[U_i]$-module M_i then $\Gamma(U_i, F|_{U_i}) = M_i$ and $\Gamma(U_i, F_K|_{U_i}) = M_i \otimes_k K$ by definition. Thus the groups $H^i(X_K, F_K)$ are computed from the above Čech-complex by first tensoring with K and then taking cohomologies.

It is basic linear algebra that the kernel and cokernel of a linear map change by tensoring with K under a field extension K/k. □

SOLUTION TO 3.40.

This is done in the solution to Exercise 3.4. □

SOLUTION TO 3.41.

This is essentially pure commutative algebra. The point is that normality is characterized locally by the two conditions R_1 (meaning regular in codimension one), and S_2, Serre's S_2 condition; see Matsumura (1980, p.125) for the definition. Since Serre's S_2 condition is satisfied automatically by every hypersurface, the result follows. See, for instance, Matsumura (1980, p.125) for proofs. □

SOLUTION TO 3.42.

(1) Let C be an irreducible and reduced member of $|-K_X|$, which exists and has arithmetic genus one by Proposition 3.39. Assuming that $K_X^2 = 2$, consider

the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(-K_X) \to \mathcal{O}_C(-K_X|_C) \to 0.$$

Since $\mathcal{O}_C(-K_X|_C)$ is a degree two line bundle on a genus one curve, it is generated by global sections by Exercise 3.45. Since $H^1(X, \mathcal{O}_X) = 0$, we conclude that $\mathcal{O}_X(-K_X)$ is generated by global sections. Thus $|-K_X|$ gives a degree two morphism $X \to \mathbb{P}^2$. In characteristic zero, or indeed, any characteristic other than two, the ramification locus has degree 4. In characteristic two, the morphism can be purely inseparable.

(2) If $K_X^2 = 1$, then a similar analysis of the sequence

$$0 \to \mathcal{O}_X(-K_X) \to \mathcal{O}_X(-2K_X) \to \mathcal{O}_C(-2K_X|_C) \to 0$$

shows that $|-2K_X|$ defines a degree two morphism to a quadric $X \to Q \subset \mathbb{P}^3$. The image of $|-K_X|$ is a base pointed pencil of degree one curves on Q. This implies that Q is singular. □

Solution to 3.44.

Set $L = \mathcal{O}_X(D)$. Working inductively, we prove that the multiplication maps

$$\alpha_m : \sum_{i=1}^{r} R_{m-i}(X, L) \otimes R_i(X, L) \to R_m(X, L)$$

are surjective for every $m \geq r + 1$. The surjectivity of all these maps is equivalent to the conclusion.

Let us consider the following diagram, where we save space by writing R_i to denote $R_i(X, L)$.

$$\begin{tikzcd}
H^0(X, L^m) \arrow[r, "s"] & H^0(X, L^{m+1}) \arrow[r, two heads, "\text{rest}"] & H^0(D, (L|_D)^{m+1}) \\
 & & \sum_{i=1}^{r} R_{m+1-i}(D, L) \otimes R_i(D, L) \arrow[u, "\gamma_{m+1}"] \\
\sum_{i=1}^{r} R_{m-i} \otimes R_i \arrow[uu, "\alpha_m"] \arrow[r, "\otimes s"] & \sum_{i=1}^{r} R_{m+1-i} \otimes R_i \arrow[uu, "\alpha_{m+1}"] \arrow[r, two heads, "\text{rest}"] & \sum_{i=1}^{r} R_{m+1-i}(D, L) \otimes R_i \arrow[u, "\text{rest}"']
\end{tikzcd}$$

Here "rest" denotes a restriction map; these are all surjective by (2). The top and bottom rows are exact. Note that γ_{m+1} is surjective by assumption (1), and working inductively we may assume that α_m is surjective. This implies that α_{m+1} is also surjective. □

SOLUTION TO 3.45.

Parts (1), (2), and (3) follow from Exercise 3.44.

To see that (4) follows from (2), consider a divisor D of degree at least two. From (2), the section ring of R of D is generated in degrees one and two. Thus the equations of members of the linear system $|mD|$ are spanned by the monomials of degree m in the degree one elements $s_1, \dots, s_p$ and the degree two elements $t_1, \dots, t_q$ of R. If D has a base point P, then all the s_i vanish at D, and hence every monomial of odd degree in the s_i and t_j vanishes at P as well. This means that for odd m, the linear system $|mD|$ has a base point at P. This is a contradiction when m is large.

The proof of (5) is similar. □

SOLUTION TO 3.46.

We may assume that everything is over an algebraically closed field. Let C be an irreducible and reduced divisor in $|-K_X|$, which exists by Proposition 3.39. To prove the statements about the degrees in which the section rings are generated, we wish to apply Exercise 3.44. The necessary vanishing follows from Lemma 3.38, and the statement about the restriction of $-K_X|_C$ is handled by Exercise 3.45. Thus we obtain the statements about the degrees in which the section rings are generated in cases (1), (2), and (3).

To see that $-K_X$ is ample in case (1), we note the following general fact: if a section ring of any ample line bundle L is generated in degree one, then L is very ample. Indeed, if $H^0(X, L)$ generates the ring $\sum H^0(X, L^i)$, then any two points (or tangent vectors) not separated by sections of L can not be separated by sections of L^i. Thus $-K_X$ is very ample for a Del Pezzo surface of degree at least three.

A similar argument shows that $-K_X$ is globally generated in case (2). More generally, if a section ring of any ample line bundle L is generated in degrees one and two, then L is globally generated. Indeed, if $H^0(X, L)$ does not separate points and the ring $\sum H^0(X, L^i)$ is generated in degrees one and two, then clearly no $H^0(X, L^i)$ with i odd can separate points. But all sufficiently high powers of an ample line bundle separate points. □

SOLUTION TO 3.51.

Note that

$$U_i \cap U_j = \operatorname{Spec} k\left[x_0, \dots, x_n, x_i^{-1}, x_j^{-1}\right]_{(0)}.$$

By assumption, there is a monomial $M = x_i^{b_i} x_j^{b_j}$ whose degree is 1. Then

$$k\left[x_0, \dots, x_n, x_i^{-1}, x_j^{-1}\right]_{(0)} = k\left[\frac{x_s}{M^{a_s}} : (s \neq i, j), \frac{x_i^{a_j}}{x_j^{a_i}}, \frac{x_j^{a_i}}{x_i^{a_j}}\right]$$
$$\cong k[u_s : (s \neq i, j), v, v^{-1}].$$

Thus in fact $U_i \cap U_j \cong \mathbb{A}^{n-1} \times (\mathbb{A}^1 \setminus \{0\})$. □

Solution to 3.53.

On the chart U_0 we have coordinates $u_i = x_i/x_0^{a_i}$ for $i = 1, \dots, n$ and on the chart U_1 we have coordinates $v_i = x_i/x_1^{a_i}$ for $i = 0, 2, \dots, n$. The transition functions are

$$u_1 = v_0^{-1} \quad \text{and} \quad u_i = v_i v_0^{-a_i} \ : \ i \geq 2.$$

$\sigma = du_1 \wedge \cdots \wedge du_n$ is a nowhere-zero section of the canonical line bundle of U_0. Since

$$du_1 \wedge \cdots \wedge du_n = -v_0^{\sum a_i} dv_0 \wedge dv_2 \wedge \cdots \wedge dv_n$$

we see that σ has a pole of order $\sum a_i$ along $(x_0 = 0)$. □

Solution to 3.55.

In our experience, computations requiring several blowups are best left to the reader. □

Solution to 3.56.

It is easy to check that $-K_{S'} \cdot C > 0$ for every curve $C \subset S'$, and that $-K_{S'}^2 = -K_S^2 + 1$. If you know the Nakai–Moishezon criterion (cf. Hartshorne 1977, V.1. 10), then these statements together already imply that $-K_{S'}$ is ample.

Alternatively, we see right away that $-K_{S'}$ is ample on $S' \setminus P$. Thus we are done using the following general result.

Proposition 7.4. *Let X be a projective variety of dimension at least two, and let L be a line bundle on X. Suppose that there is some finite set Z of X such that $L|_{X \setminus Z}$ is ample. Then L is ample on X.*

Proof. Pick a section of L^k with zero set $Y \subset X$. If $\dim X \geq 3$ then $L|_Y$ is ample by induction on the dimension. If $\dim X = 2$ then Y is a curve. Pick points $y_i \in Y \setminus Z$. For some $m \gg 1$, L^m has a section which is zero at the points y_i but does not vanish on (any irreducible component of) Y. Thus $L^m|_Y$ is a line bundle whose degree can be as large as we want, thus again L_Y is ample. Consider now the sequence

$$0 \to L^{mk-k} \to L^{mk} \to L^{mk}|_Y \to 0.$$

If $m \geq m_0$ then $H^1(Y, L^{mk}|_Y) = 0$, hence we get surjections

$$H^1(X, L^{mk-k}) \twoheadrightarrow H^1(X, L^{mk}) \twoheadrightarrow H^1(X, L^{mk+k}) \twoheadrightarrow \cdots$$

Eventually the surjections become isomorphisms for some $m \geq m_1$. Thus

$$H^0(X, L^{mk}) \twoheadrightarrow H^0(Y, L^{mk}|_Y) \quad \text{for } m \geq m_1.$$

This implies that L^{mk} is generated by its global sections and the rest is easy. □

The rest of the proof of Exercise 3.56 follows from Theorem 3.5. □

7.4 Exercises in Chapter 4

SOLUTION TO 4.8.

Set $\deg f = e > d + 1$. The hypersurface in $\mathbb{P}^{n+1}$ is given by a homogeneous polynomial

$$G = y^d t^{e-d} - (F_0 t^e + F_1 t^{e-1} \cdots + F_{e-1} t + F_e),$$

where each F_i is a homogeneous polynomial of degree i in $x_1, \ldots, x_n$. The nonsmooth locus is defined by the homogeneous ideal

$$\left(G, \frac{\partial G}{\partial t}, \frac{\partial G}{\partial y}, \frac{\partial G}{\partial x_1}, \ldots, \frac{\partial G}{\partial x_n} \right).$$

Note that each derivative above is contained in the ideal $(t, x_1, \ldots, x_n)$. So the nonsmooth locus contains the point $(y, x_1, x_2, \ldots, x_n, t) = (1, 0, \ldots, 0)$, regardless of the characteristic. □

SOLUTION TO 4.9.

Regardless of the characteristic of the ground field, the nonsmooth locus of an affine hypersurface defined by G is the locus defined by G and all its partial derivatives.

In particular, the nonsmooth locus of the hypersurface defined by $y^p - f$ is the closed set defined by the ideal generated by $y^p - f$, py^{p-1} and the partial derivatives of f. In characteristic zero, therefore, any nonsmooth point will have y coordinate zero. So a nonsmooth point has the form $(y, x_1, \ldots, x_n) = (0, \lambda_1, \ldots, \lambda_n)$, where all the partial derivatives of $f(x_1, \ldots, x_n)$ vanish at $(\lambda_1, \ldots, \lambda_n)$. Since $y^p - f = 0$, it must be that $(\lambda_1, \ldots, \lambda_n)$ is a critical point of critical value zero. Equivalently, $(\lambda_1, \ldots, \lambda_n)$ is a nonsmooth point of the hypersurface in n-space defined by f. But a sufficiently general polynomial f defines a smooth hypersurface, so in characteristic zero a general hypersurface of the form $y^p - f$ to be smooth.

In characteristic p, the derivative with respect to y vanishes, so the nonsmooth locus is defined by the ideal $(y^p - f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$. Every critical point $(\lambda_1, \dots, \lambda_n)$ of f determines exactly one nonsmooth point, by setting $y_i = f(\lambda)^{1/p}$. (Of course, f could fail to have critical points at all, as in the example $f = x_1 + x_2^{mp}$, but this means simply that the critical points are hiding at infinity.) For a general f, the expected dimension of the locus where all the $\frac{\partial f}{\partial x_i}$ vanish is zero. Thus a general hypersurface of the form $y^p - f$ in characteristic p has only isolated nonsmooth points. □

Solution to 4.16.

Consider a connection $\nabla : \mathcal{L} \to \mathcal{L} \otimes \Omega_X$ on X. On an open set U where $\mathcal{L}$ is trivial, fix an isomorphism $\mathcal{O}_X(U) \cong \mathcal{L}(U)$, with $g \in \mathcal{L}$ corresponding to 1 in $\mathcal{O}_X$. Set $\nabla(g) = g \otimes \eta$, for some one-form $\eta \in \Omega_X(U)$. On U, we have

$$\mathcal{L} \xrightarrow{\nabla} \mathcal{L} \otimes \Omega_X$$
$$fg \mapsto f\nabla(g) = g \otimes df = g \otimes (df + f\eta).$$

So we can think of ∇ as a gadget that associates to the local section f, the one-form $df + f\eta$.

Now, we can "differentiate a section of $\mathcal{L}$ in any tangent direction." Indeed, a tangent direction is interpreted as a section of the sheaf of derivations of $\mathcal{O}_X$ to $\mathcal{O}_X$. Since $\mathrm{Der}(\mathcal{O}_X, \mathcal{O}_X) = \mathrm{Hom}(\Omega_X, \mathcal{O}_X)$, each derivation θ produces a homomorphism $\theta : \Omega_X \to \mathcal{O}_X$. Its value on the one form $df + f\eta$ can be considered the derivative of f in the direction of θ. So the local section fg of $\mathcal{L}$ is sent to the section $\theta(df + f\eta)\,g$ of $\mathcal{L}$.

For any line bundle $\mathcal{L}$, we can naively try to differentiate local sections as follows. Over an open set U where $\mathcal{L}$ is trivial with fixed generator g, if we are given a section θ of the sheaf of derivations, we try sending each section $f \cdot g$ of $\mathcal{O}_X \cdot g$ to $(\theta f) \cdot g \in \mathcal{O}_X \cdot g \cong \mathcal{L}$. In general, of course, this does not lead to a globally well defined method for differentiating sections of $\mathcal{L}$, because patching fails. Indeed, let g_1 and g_2 be local generators for $\mathcal{L}$, related by the transition function $g_1 = \phi g_2$. If s is a local section of $\mathcal{L}$, then writing $s = fg_1 = (\phi f)g_2$, we see that $\theta(s)$ is well defined if and only if $\theta(\phi f) = \phi\theta(f)$. Using the Leibniz rule for derivations, we see that this is equivalent to $\theta(\phi) = 0$. Thus, this naive approach to differentiating sections gives a well defined global connection on $\mathcal{L}$ if and only if $\mathcal{L}$ admits transition functions that are killed by all derivations. Because derivations annihilate any function that is a pth power, it follows that any line bundle $\mathcal{L}$ that is a pth power of another line bundle $\mathcal{M}$ admits this connection, as the transition functions for $\mathcal{L}$ can be taken to be pth powers of transition functions for $\mathcal{M}$.

Let $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(mp)$ and fix a global section f of $\mathcal{L}$. A convenient choice of local trivialization for $\mathcal{L}$ is to let U_i be the set where the homogeneous coordinate x_i does not vanish. On U_i, we can consider x_i^{mp} to be a local generator. Thinking of f as a homogeneous polynomial of degree mp in the homogeneous coordinates for $\mathbb{P}^n$, it has representation $(f/x_i^{mp})x_i^{mp}$ on U_i. $\square$

SOLUTION TO 4.20.

The ground field may be assumed algebraically closed. We check the case where n is even; the case where n is odd is similar. At an isolated nonsmooth point, after making suitable linear changes of coordinates, we can assume the equation has the form

$$y^p - x_1x_2 - x_3x_4 - \cdots - x_{n-1}x_n - f_3(x_1, \ldots, x_n).$$

Blowing up the ideal generated by $y, x_1, \ldots, x_n$, the resulting scheme is covered by affine patches where $x_i \neq 0$ or $y \neq 0$.

Consider first the case $y \neq 0$. There are local coordinates $y, x_1', x_2', \ldots, x_n'$ where $yx_i' = x_i$ for $i \geq 1$. The blown up hypersurface is defined by

$$y^{p-2} - x_1'x_2' - x_3'x_4' - \cdots - x_{n-1}'x_n' - f_1'(y, x_1', x_2', \ldots, x_n').$$

Repeating, we eventually get down to $p - 2 = 0$ or 1, in which case the singularities of the hypersurface are resolved.

The other patches are all alike, consider say where $x_1 \neq 0$. There are local coordinates $y', x_1, x_2', x_3', \ldots, x_n'$ where $x_1y' = y$, and $x_1x_i' = x_i$ for $i > 1$. The blown up hypersurface is defined by

$$x_1{}^{p-2}y'^p - x_2' - x_3'x_4' - \cdots - x_{n-1}'x_n' - x_1f_1'(x_1, x_2', \ldots, x_n')$$

where f_1' has order at least one in $(x_1, x_2', \ldots, x_{2n}')$. This hypersurface is easily verified to be smooth, using the Jacobian criterion. Alternatively, it is sufficient to check that the exceptional divisor, namely the divisor defined by $x_1 = 0$ on this hypersurface, is smooth. This is obvious, since its equation is $x_2' - x_3'x_4' - \cdots - x_{n-1}'x_n'$ (or $y'^p - x_2' - x_3'x_4' - \cdots - x_{n-1}'x_n'$ when $p = 2$). $\square$

SOLUTION TO 4.22.

Let $x_1, \ldots, x_n$ be local coordinates around P. In these coordinates, polynomials with a critical point at P all have the form

$$f(x_1, \ldots, x_n) = a + \sum_{i \leq j} a_{ij}x_ix_j + \text{ (higher order terms)}$$

where $a_{ij} \in k$ (by considering a Taylor series expansion, for instance). The

Hessian of f is the symmetric matrix

$$A = \begin{pmatrix} 2a_{11} & a_{12} & \dots & a_{1n} \\ a_{12} & 2a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & 2a_{nn} \end{pmatrix}$$

and the invertibility of A is equivalent to the non-vanishing of the determinant of this symmetric matrix. (It is easy to verify that this is also equivalent to the condition that the $\frac{\partial f}{\partial x_i}$s generate the maximal ideal $(x_1, \dots, x_n)$ of P.)

In any characteristic other than two, the determinant of a symmetric matrix is a nonzero polynomial in the entries. (For instance, the coefficient of $a_{11}a_{22}\cdots a_{nn}$ is $2^n \neq 0$.) Therefore, on a Zariski open subset of the finite dimensional vector space of quadratic polynomials in the x_i, the coefficient matrix (a_{ij}) has nonzero determinant. Thus the subset of all polynomials of degree $d \geq 2$ which have a critical point at P is defined by the following $n+1$ independent equations:

1. the linear part of the Taylor expansion (n equations), and
2. the Hessian of the quadratic part.

Varying the point P gives an n-dimensional family, so polynomials with at least one degenerate critical point form a subset of codimension at least $n + 1 - n = 1$.

Therefore a "sufficiently general" polynomial has only non-degenerate critical points, assuming the characteristic is not two.

In characteristic two, however, a symmetric matrix is also a skew-symmetric matrix. In n is odd, it always has determinant zero, so all critical points of f are degenerate in this case. If n is even, however, the determinant of a symmetric $n \times n$ matrix is a nonzero polynomial, and again we conclude that a generic f in an even number of variable has non-degenerate critical points. See Jacobson (1985, pp. 332–335) for these basic facts on skew-symmetric (or alternating) forms, or convince yourself by looking at the cases $n \leq 4$. □

Solution to 4.25.

This is pure commutative algebra. Recall that a local ring is regular of dimension d if and only if its maximal ideal can be generated by d elements. If the local ring R/u is regular of dimension $d-1$, then its maximal ideal can be generated by $d-1$ elements $x_1, \dots, x_{d-1}$. Lifting these elements to R, we see that $u, x_1, \dots, x_{d-1}$ generate the maximal ideal of R. Since u is a nonzero divisor, the ring R has dimension d, and so it is regular. □

SOLUTION TO 4.28.

The question is local, so assume $Z \to Y \to X$ are maps of affine schemes corresponding to the ring maps $A \to B \to C$. By our finite type assumption, we know B is a finitely generated A-algebra with generators, say $x_1, \ldots, x_n$ and relations, say $f_1, \ldots, f_n$. We can assume the same number of generators and relations because both A and B are regular of the same dimension. Likewise, C is a finitely generated B algebra with generators $y_1, \ldots, y_m$ and relations, say $g_1, \ldots, g_m$. Thus $x_1, \ldots, x_n, y_1, \ldots, y_m$ are A algebra generators for C, with relations $f_1, \ldots, f_n, g_1, \ldots, g_m$.

In this case, the Jacobian ideal for B over A is the principal ideal given by the determinant of the $n \times n$ Jacobian matrix

$$\left(\frac{\partial f_i}{\partial x_j}\right)$$

and the Jacobian ideal for C over B is given by the determinant of the $m \times m$ matrix

$$\left(\frac{\partial g_i}{\partial y_j}\right).$$

Because the Jacobian ideal of C over A is the determinant of the $(m+n) \times (m+n)$ matrix

$$\begin{pmatrix} \frac{\partial f_i}{\partial x_j} & 0 \\ \frac{\partial g_i}{\partial x_j} & \frac{\partial g_i}{\partial y_j} \end{pmatrix}$$

(in block form), the result is immediate. □

7.5 Exercises in Chapter 5

SOLUTION TO 5.4.

Let $f : T \to S$ be the blow up of a point P in S. Then

$$K_T = f^* K_S + E \quad \text{and} \quad f_*^{-1} H = f^* H - m_P E$$

where $E \subset T$ is the exceptional curve and m_P is the multiplicity of H at P. Thus for any positive integer m, we have

$$K_T + \tfrac{1}{m} f_*^{-1} H \equiv f^*(K_S + \tfrac{1}{m} H) + (1 - \tfrac{m_P}{m}) E, \tag{5.4.1}$$

and the divisor $E_f - \frac{1}{m} F_f$ is effective if and only if $1 - \frac{m_P}{m}$ is non-negative.

Now assume (1). The above formula for $g = f$ gives that $1 - \frac{m_P}{m} \geq 0$, that is, the multiplicity m_P of H at P is at most m.

Conversely, assume (2). Given any proper birational map $g : S' \to S$ from a normal S, we resolve the singularities of S' to get a diagram

$$\begin{tikzcd} S_n \arrow[r, "f_n"'] \arrow[rrrr, bend left, "h"] & \cdots \arrow[r, "f_2"'] & S_1 \arrow[r, "f_1"'] & S_0 = S & S' \arrow[l, "g"] \end{tikzcd}$$

where each S_i is smooth and obtained from S_{i-1} by blowing up a point. Let f be the composition map $S_n \to S$ and note that the image of the divisor $E_f - \frac{1}{m}F_f$ on S' is precisely the divisor $E_g - \frac{1}{m}F_g$. Thus it suffices to prove that $E_f - \frac{1}{m}F_f$ is effective.

We do this by induction on n, the number of blowups. We factor f as

$$S_n \xrightarrow{f'} S_1 \xrightarrow{f_1} S$$

where f_1 is the blowup of some point P in S. The computation in the first paragraph shows that the result holds for $n = 1$. The inductive step implies that

$$K_{S'} + \tfrac{1}{m}(f_n)_*^{-1}H \equiv (f')^*(K_{S_1} + \tfrac{1}{m}\left(f_1^{-1}\right)_* H) + (\text{effective divisor}).$$

Combining this with the formula (5.4.1) for $f = f_1$, we get

$$K_{S_n} + \tfrac{1}{m}(f_n^{-1})_* H \equiv f_n^*(K_S + \tfrac{1}{m}H) + (\text{effective divisor}).$$

This completes the proof. □

SOLUTION TO 5.6.

We may assume that X admits a very ample divisor M whose birational transform M' on X' is also very ample. Let $Z \subset X \times X'$ be the closure of the graph of ϕ with projections π and π'. The linear systems $M' = \phi_* M$ and $\pi'_*\pi^* M$ agree outside the codimension two (or more) set W', so $\phi_* M = \pi'_*\pi^* M$ as divisors on X'. Assume that there is a curve $C \subset Z$ which is contained in a fiber of π'. Then $\pi(C)$ is a curve and so every member of $\pi^* M$ intersects C. This implies that $\pi'(C)$ is a base point of $M' = \phi_* M$. This is impossible since M' is very ample. Therefore, π' is finite, and hence an isomorphism. By reversing the roles of X and X', we see that π is also an isomorphism, and thus so is ϕ. □

SOLUTION TO 5.17.

Note that the divisor W defined by the vanishing of w is the only exceptional divisor of g. We compute $g^*(dx \wedge dy \wedge dz) = w^{a+b} du \wedge dv \wedge dw$, so that $E_g = (a+b)W$. Since $g^*(x^i y^j z^k)$ vanishes along W with multiplicity $ai + bj + k$, we see that F_g contains W with multiplicity greater than $m(a+b)$. Therefore $E_g - \frac{1}{m} F_g$ is not effective. □

SOLUTION TO 5.21.

The intersection number can be computed in the local coordinates given by Theorem 5.19. We write the chart as $\mathbb{A}^3$ though it is best to think of it as a small Euclidean neighborhood of the origin P in $\mathbb{C}^3$.

Consider the map $q : \mathbb{A}^3_{(u,v,w)} \to \mathbb{A}^3_{(x,y,z)}$ given by $x = u^a$, $y = v^b$, $z = w$. Then q has degree ab, hence

$$(q^* H_1 \cdot q^* H_2 \cdot q^* S)_P = ab(H_1 \cdot H_2 \cdot S)_P.$$

On the other hand, $q^* H_i$ (respectively, $q^* S$) have multiplicity $> m(a+b)$ (respectively, > 1), hence

$$(q^* H_1 \cdot q^* H_2 \cdot q^* S)_P > m^2(a+b)^2.$$

Dividing by ab and using $(a+b)^2/ab \geq 4$ gives the result. □

SOLUTION TO 5.23.

Let D be a very ample linear system on X and choose $n-1$ general members $D_1, \dots, D_{n-1}$. Then $C = D_1 \cap \dots \cap D_{n-1}$ is a smooth curve and by the adjunction formula its canonical class is $K_X + (n-1)D|_C$. Thus

$$2g(C) - 2 = (K_X + (n-1)D) \cdot D^{n-1}.$$

We can take $D = |K_X + 2H|$ for some ample divisor H. Computing modulo 2 we get that

$$2g(C) - 2 = (K_X + (n-1)(K_X + 2H)) \cdot (K_X + 2H)^{n-1} \equiv n \cdot K_X^n \mod 2.$$

If n is odd, this implies that K_X^n is even. If n is even, K_X^n can be even or odd. □

7.6 Exercises in Chapter 6

SOLUTION TO 6.5.

From formula (6.3.3) we know that there is an f-exceptional divisor F such that

$$K_Y + f_*^{-1}\Delta + F \equiv f^*(K_X + \Delta).$$

By construction $f_*(f_*^{-1}\Delta + F) = f_*(f_*^{-1}\Delta) = \Delta$. Thus it is enough to prove that $\Delta_Y = f_*^{-1}\Delta + F$. Since $f_*\Delta_Y = \Delta$, we see that $f_*^{-1}\Delta = f_*^{-1}(f_*\Delta_Y)$ is a summand of Δ_Y. Thus we can write $\Delta_Y = f_*^{-1}\Delta + F_Y$ where $f_* F_Y = 0$, and hence F_Y is f-exceptional. Comparing the condition (2) to the above formula we conclude that

1. $F_Y - F$ is f-exceptional, and
2. $F_Y - F$ is numerically trivial.

We are going to conclude that $F_Y - F = 0$, which then gives exactly what we want.

Let H_Y be a very ample linear system on Y and let $S \subset Y$ be the intersection of $\dim Y - 2$ general members of H_Y. Let E denote the restriction of $F_Y - F$ to S. Then $E = 0$ if and only if $F_Y - F = 0$, E is numerically trivial and E is exceptional with respect to $f|_S : S \to f(S) \subset X$.

Let H_X be ample on X and set $H_S := f^*H_X|_S$. Then $H_S \cdot H_S > 0$ and H_S has zero intersection number with every irreducible component of E. Thus by the Hodge index theorem (see Hartshorne, 1977, V. 1.9.1), the intersection product on the irreducible components of E is negative definite, hence $E \equiv 0$ implies that $E = 0$. (The form of the Hodge index theorem proved in Hartshorne (1977, V.1.9.1) applies to smooth surfaces only, so, to be prudent, it should be applied to a resolution of singularities $g : S' \to S$ and to the pull-back g^*E. The latter exists since we also know that $F_Y - F$ is $\mathbb{Q}$-Cartier.) □

SOLUTION TO 6.7.

It is enough to consider the case where $\Delta' = aD$ where D is a linear system (possibly consisting only of fixed components) and $a > 0$.

Fix a birational morphism $f : Y \to X$ from a normal variety so that E is an exceptional divisor for f. First note that $a(f^*D - f_*^{-1}D)$ has only non-negative coefficients, so (1) is clear from the expression (6.1.3).

Strict inequality occurs if and only if E appears in the effective exceptional $\mathbb{Q}$-divisor $f^*D - f_*^{-1}D$ with nonzero coefficient. If E does appear, then the center of E on X is contained in every member of D. Conversely, if the center of E on X is contained in the base locus of D, then E is contained in every member of f^*D. But no exceptional divisor is a fixed component of $f_*^{-1}D$, since its fixed components are precisely the birational transforms of the fixed components of D (see §5.1). Thus the exceptional divisor E appears in $f^*D - f_*^{-1}D$ if any only if the center of E on X is contained in the base locus of D. This completes the proof. □

SOLUTION TO 6.9. Let F be a divisor over X. If F lies on Y, then the equality $a(F, Y, \Delta_Y) = a(F, X, \Delta_X)$ is immediate from the definition of Δ_Y. We therefore assume that F does not appear as a divisor on Y. Without loss of generality, we may assume that F lies on a normal variety Z which admits a birational morphism $g : Z \to Y$ to Y (for example, we may take Z to be the normalization of the graph of the map $Y \dashrightarrow X'$ where $X' \to X$ is any birational morphism to X for which F is an exceptional divisor; see Remark 6.6). To compute the discrepancy $a(F, X, \Delta)$, we find Δ_Z such that

$$K_Z + \Delta_Z \equiv g^*(K_Y + \Delta_Y) \quad \text{and} \quad g_*\Delta_Z = \Delta_Y.$$

Since

$$K_Y + \Delta_Y \equiv f^*(K_X + \Delta_X) \quad \text{and} \quad f_*\Delta_Y = \Delta_X,$$

we also conclude that

$$K_Z + \Delta_Z \equiv (f \circ g)^*(K_X + \Delta_X) \quad \text{and} \quad (f \circ g)_*\Delta_Z = \Delta_X.$$

Thus the multiplicity of F in Δ_Z is $-a(F, X, \Delta)$ by the first formula and also $-a(F, Y, \Delta_Y)$ by the third formula. Thus $a(F, X, \Delta) = a(F, Y, \Delta_Y)$. □

SOLUTION TO 6.10.
Simply compute

$$K_Y - p^*K_X = (c-1)E \quad \text{and} \quad f_*^{-1}D_i = f^*D_i - (\mathrm{mult}_Z\, D_i)E,$$

for each i. The result follows immediately. □

SOLUTION TO 6.13. (1). Suppose that for some divisor F exceptional over X, the discrepancy $a(F, X, \Delta)$ is $-1-c$ for some positive number c. Without loss of generality, we may assume that F lies on a smooth variety Y admitting a birational morphism to X, since we can replace Y by its dense open set of smooth points (which still contains the generic point of F). Blowing up a sufficiently general codimension one subvariety of F (which is thus codimension two in the smooth variety Y), Exercise 6.10 implies that we obtain an exceptional divisor E such that $a(E, X, \Delta) = -c$. Now blowing up the intersection of the birational transform of F and E, the discrepancy along the new exceptional divisor is $-2c$. By repeatedly blowing up the intersection of the birational transform of F with the most recent exceptional divisor, one checks that the discrepancy of the new exceptional divisor at the mth step is $-mc$, so the discrepancy of (X, Δ) is $-\infty$.

(2) Without loss of generality, we may assume that X is smooth. Indeed, if X^0 denotes the dense open set of smooth points of X, then clearly $\mathrm{discrep}(X, \Delta) \leq \mathrm{discrep}(X^0, \Delta|_{X^0})$. Now let Z be any codimension two subvariety of X not contained in the base locus of any of the linear systems making up Δ. If $f : Y \to X$ denotes the blowup of Z, then the exceptional divisor E does not appear in the exceptional divisor $f^*(D_i) - f_*^{-1}(D_i)$ for any i. But E_i appears in

$K_Y - f^*K_X$ with multiplicity exactly one. Thus $a(E, X, \Delta) = 1$, giving the desired bound. □

SOLUTION TO 6.14. By Exercise 6.13, the discrepancy is bounded above by one, so we need only show that it is always a positive integer.

Let $f : Y \to X$ be a birational morphism and $E \subset Y$ an exceptional divisor. Let $p \in E \subset Y$ be a point of the exceptional set which is smooth on Y and on $\mathrm{Ex}(f)$. Let y_i (respectively x_i) be local coordinates at p (respectively $f(p)$). f is given by local coordinate functions $x_i = f_i(y_1, \ldots, y_n)$. Then

$$f^*dx_1 \wedge \cdots \wedge dx_n = \mathrm{Jac}\left(\frac{f_1, \ldots, f_n}{y_1, \ldots, y_n}\right) dy_1 \wedge \cdots \wedge dy_n,$$

where Jac is the Jacobian. Since p is on the exceptional set, f has no local inverse at p and so $\mathrm{Jac}(p) = 0$ by the inverse function theorem. The discrepancy of E is the order of vanishing of Jac along E, which is thus positive. The proof is complete. □

SOLUTION TO 6.15. This follows immediately from Exercise 6.9 and the definition of discrepancy. Note that the definition of discrepancy of the pair (Y, Δ_Y) only involves exceptional divisors over Y, which is why the divisors E exceptional over X but not Y must be considered separately. □

SOLUTION TO 6.18. The proof is exactly the same as the proof of Proposition 5.11. □

SOLUTION TO 6.22. Let E be an exceptional divisor over X. We may assume that E lies on a smooth variety Y admitting a birational morphism $f : Y \to X$ which factors as a sequence of blowups at smooth centers. (As we have seen, this is transparent using resolution of singularities, but resolution is not really needed; see Remarks 4.27 and 5.13.)

Let us consider what happens after one blowup. Assume that $f : Y \to X$ is the blowup of a smooth codimension c subvariety V and that E is the resulting exceptional divisor. Since each component D_i is smooth, the coefficient of E in

$$f^*D_i - f_*^{-1}D_i$$

is zero or one, depending on whether or not the center V of the blowup is contained in D_i. Since also $K_Y - f^*K_X = (c-1)E$, we have

$$a(E, X, \Delta) = (c-1) - \sum_{V \subset D_i} a_i, \tag{6.22.1}$$

where the sum is taken over all components D_i containing V. Since the D_i intersect transversely, at most c of them can contain the codimension c subvariety V. The desired discrepancy bounds needed in statements (1)–(4) all follow. For example, if each $a_i \leq 1$, then the discrepancy along E is at least -1. Similarly, if $a_i + a_j \leq 1$ whenever D_i and D_j intersect, then the discrepancy along E is at least zero.

To see what happens after several blowups, note that by Exercise 6.15, we can replace the pair (X, Δ) by (Y, Δ_Y), where

$$\Delta_Y = f_*^{-1}\Delta - a(E, X, \Delta)$$

(with f and E as in the preceding paragraph). The divisor Δ_Y satisfies the same hypothesis as Δ, and so the same argument shows that its discrepancy along the exceptional fiber of the next blowup at a smooth center is bounded as before. By induction, the proof is complete. □

Solution to 6.23. This follows easily from the previous exercise, by virtue of Exercise 6.15. We have

$$\operatorname{discrep}(X, \Delta) = \min\{\operatorname{discrep}(Y, \Delta'_Y), a(E, X, \Delta)\}$$

where E runs through all the exceptional divisors of the log resolution f.

If (X, Δ) is log canonical, then clearly so is (Y, Δ'_Y), so $e_i \leq 1$ for all i by Exercise 6.22. If (X, Δ) is furthermore canonical, then also the discrepancies $a(E, X, \Delta_X)$ are non-negative for f-exceptional divisors E.

Conversely, if the stated conditions on the e_i hold, then discrep(Y, Δ'_Y) is greater than -1 (respectively 0) by Exercise 6.22. But for exceptional D_i we have $a(D_i, X, \Delta) = -e_i$. Thus if the e_i satisfy the stated conditions, then the pair (X, Δ) is log canonical (respectively, canonical). The arguments for plt and klt are similar. □

Solution to 6.24. Choose a log-resolution $f : Y \to X$ and write

$$K_Y + cf_*^{-1}H - \sum a_i E_i \equiv f^*(K_X + cH),$$

where $f_*^{-1}H$ is free and $a_i = a(E_i, X, \Delta)$. (X, cH) is canonical if and only if $a_i \geq 0$. Let $D \in H$ be a general member with birational transform $D_Y \in f_*^{-1}H$. Then $D_Y + \sum E_i$ is a simple normal crossing divisor. If $c \leq 1$, the conditions of Exercise 6.22.2 are satisfied (since D_Y is the only divisor with positive coefficient) and (X, cD) is canonical.

If $c > 1$ then (X, cD) is not even log canonical, so in this case we replace D by $\frac{1}{m}(D_1 + \cdots + D_m)$. In general the birational transforms of D_i and D_j may intersect, so we also have to be mindful of the condition that the sum of the coefficients of two intersecting divisors be at most 1. Thus we need to ensure that $\frac{c}{m} \leq \frac{1}{2}$. $\square$

SOLUTION TO 6.26. Let U be the open subset of X consisting of all smooth points where g is not ramified. (By the "purity of branch loci" g can not ramify over a smooth point, but we do not need this.) Set $U' = g^{-1}(U)$. Then $g : U' \to U$ is finite and unramified, and

$$K_{U'} + g^*\Delta|_{U'} \sim g^*(K_U + \Delta|_U).$$

If $m(K_X + \Delta)$ is Cartier, then $g^*(m(K_X + \Delta))$ is a Cartier divisor which agrees with $m(K_{U'} + g^*\Delta|_{U'})$ over U'. Thus

$$g^*(m(K_X + \Delta)) \sim m(K_{X'} + g^*\Delta).$$

This shows that $m(K'_X + g^*\Delta)$ is Cartier.

In order to prove (1), let $f : Y \to X$ be a birational morphism and consider the following commutative diagram of normal varieties

$$\begin{array}{ccccc} E' & \subset & Y' & \stackrel{f'}{\to} & X' \\ \downarrow & & \downarrow h & & \downarrow g \\ E & \subset & Y & \stackrel{f}{\to} & X, \end{array}$$

where Y' is the normalization of a component of the product $Y \times_X X'$, and E' is an f'-exceptional divisor mapping to the f-exceptional divisor E.

Let $r \leq \deg h = \deg g$ be the ramification index of h along E'. Let us compute the discrepancy along E'. In a neighborhood of E', we have

$$\begin{aligned} K_{Y'} &= f'^*(K_{X'} + g^*\Delta) + a(E', X', g^*\Delta)E' \\ &= f'^*g^*(K_X + \Delta) + a(E', X', g^*\Delta)E' \\ &= h^*f^*(K_X + \Delta) + a(E', X', g^*\Delta)E', \quad \text{and} \\ K_{Y'} &= h^*K_Y + (r-1)E' \\ &= h^*f^*(K_X + \Delta) + a(E, X, \Delta)h^*E + (r-1)E' \\ &= h^*f^*(K_X + \Delta) + (ra(E, X, \Delta) + (r-1))E'. \end{aligned}$$

This shows that $a(E', X', g^*\Delta) + 1 = r(a(E, X, \Delta) + 1)$. This implies (1).

We also get (2) if we know that every divisor E' over X' appears in some diagram as above. This is not hard to show assuming resolution of singularities, or even just the method described in §4.29.

Finally (2) implies (3). $\square$

SOLUTION TO 6.27.

The first two claims are easy, so let us assume that $(a, n) = (b, n) = 1$. This means that for every $i = 1, \ldots, n-1$ there is a unique $0 < c(i) < n$ such that n divides $ia + c(i)b$. Thus

$$R(n, a, b) \supset \mathbb{C}[x^n, y^n, x^i y^{c(i)} : i = 1, \ldots, n-1]$$

and it is clear that in fact equality holds.

Note that x, y are both integral over $R(n, a, b)$, as shown by the equation $x^n - x^n = 0$. Galois theory tells us that the degree of the quotient map is n.

In order to see the ramification points, we compare $\mathbb{C}[x, y] \supset R(n, a, b)$ with the extension $\mathbb{C}[x, y] \supset \mathbb{C}[xy^{c(1)}, y^n]$. The latter is unramified outside $y = 0$. The nice relationship is that

$$R(n, a, b)[y^{-n}] = \mathbb{C}[xy^{c(1)}, y^n][y^{-n}],$$

thus $\mathbb{C}[x, y] \supset R(n, a, b)$ is unramified outside $y = 0$. Reversing the roles of x, y we get that the only ramification is at the origin. The Hurwitz formula now gives (5).

In order to get explicit examples, note that if $i + j < n$ then $c(i + j) = c(i) + c(j)$ or $c(i + j) = c(i) + c(j) - n$. In the former case $x^{i+j} y^{c(i+j)}$ is not needed as a generator. This way we get that

$$R(n, 1, n-1) = \mathbb{C}[x^n, y^n, xy] \cong \mathbb{C}[u, v, w]/(uv - w^n).$$

Similarly

$$\begin{aligned} R(2n+1, 1, n) &= \mathbb{C}[x^{2n+1}, y^{2n+1}, xy^2, x^{n+1}y] \\ &\cong \mathbb{C}[u, v, s, t]/(s^{2n+1} - uv^2, t^{2n+1} - u^{n+1}v, s^n t - uv). \end{aligned}$$

$R(n, 1, 1)$ needs all $n + 1$ generators and it is the homogeneous coordinate ring of the degree n rational normal curve. It can be written as a quotient of $\mathbb{C}[u_0, \ldots, u_n]$ modulo relations which can best be presented in determinantal form as

$$\operatorname{rank} \begin{pmatrix} u_0 & u_1 & \cdots & u_{n-1} \\ u_1 & u_2 & \cdots & u_n \end{pmatrix} \leq 1.$$

For clarity we explain the case of the twisted cubic.

Order the monomials in $S = k[u_0, \ldots, u_3]$ lexicographically, that is,

$$u_0^{a_0} u_1^{a_1} u_2^{a_2} u_3^{a_3} > u_0^{b_0} u_1^{b_1} u_2^{b_2} u_3^{b_3}$$

if $a_0 > b_0$, or $a_0 = b_0$ and $a_1 > b_1$, etc. For $f \in S$, let $\bar{f}$ denote the leading monomial of f.

Given polynomials g and $f_1, \dots, f_k$, there are polynomials h_i and r (not necessarily unique) such that

$$g = r + \sum h_i f_i$$

and if $m \in r$ is any monomial of r, then none of the $\bar{f}_i$ divide r. This is the division algorithm for polynomials of several variables.

From now on we work with $f_1 = u_0u_2 - u_1^2$, $f_2 = u_0u_3 - u_1u_2$, $f_3 = u_1u_3 - u_2^2$ with leading monomials $m_1 = u_0u_2, m_2 = u_0u_3, m_3 = u_1u_3$. A degree d monomial $m \in S_d$ is not divisible by any of the m_i if and only if m is one of the following monomials

1. $u_0^a u_1^{d-a}$ and $a > 0$, or
2. $u_1^a u_2^{d-a}$ and $a > 0$, or
3. $u_2^a u_3^{d-a}$.

Note that there are $3d + 1$ such monomials.

Let $g \in S_d$ be a polynomial of degree d. By the previous two remarks, g is congruent modulo $I = (f_1, f_2, f_3)$ to a polynomial g_0 which is the sum of monomials of type (1–3). Upon plugging $u_0 = t_0^3$, $u_1 = t_0^2 t_1$, $u_2 = t_0 t_1^2$ and $u_3 = t_1^3$ the monomials of type (1–3) evaluate to $t_0^{3d}, t_0^{3d-1}t_1, \dots, t_1^{3d}$. Thus we get an isomorphism between S/I and $k[t_0, t_1]_{(0 \mod 3)}$, the ring generated by all monomials whose degree is divisible by 3. □

SOLUTION TO 6.31. Let X be a smooth surface, $C \subset X$ a smooth curve, Δ an effective $\mathbb{Q}$-divisor and $P \in C$ a point such that the local intersection number $(C \cdot \Delta)_P \le 1$. We need to prove that $(X, C + \Delta)$ is log canonical near P.

We prove this by induction on the number of blowups needed to get a log resolution of $(X, C + \Delta)$.

Let $f : X' \to X$ be the blowup of P, E the exceptional curve and C', Δ' the birational transforms of C, Δ. Set $m = \operatorname{mult}_P \Delta$ and note that $m \le (C \cdot \Delta)_P \le 1$. Then

$$K_{X'} + C' + \Delta' + mE \equiv f^*(K_X + C + \Delta),$$

hence by Exercise 6.9 it is enough to prove that $(X', C' + \Delta' + mE)$ is log canonical.

Let $Q \in E$ be a point not on C'. The intersection number $(E \cdot \Delta')$ equals m, hence the local intersection number satisfies $(E \cdot \Delta')_Q \le m$. Thus $(X', E + \Delta')$ is log canonical at Q by induction, and so is $(X', C' + \Delta' + mE)$ by Exercise 6.7. (C' does not matter since it does not pass through Q.)

If R is the (unique) intersection point of E and C', then

$$(C' \cdot \Delta' + mE)_R = (C' \cdot \Delta')_R + m = (C \cdot \Delta)_P - m + m = (C \cdot \Delta)_P \leq 1,$$

hence $(X', C' + \Delta' + mE)$ is log canonical at R by induction. □

SOLUTION TO 6.33. We use Exercise 6.7. First note that because Δ is effective and c is positive,

$$a(E, X, \Delta + S) = a(E, X, S) = a(E, X, c\Delta)$$

for all exceptional divisors E on X whose center on X is not contained in the union of the supports of the base loci of the components of Δ. Because X and S are both smooth, then, these discrepancies are all greater than -1.

For exceptional divisors whose center on X is contained in this union, we have

$$a(E, X, S + \Delta) = a(E, X, S + c\,\Delta + (1 - c)\Delta) < a(E, X, S + c\,\Delta).$$

So if $(X, S + \Delta)$ is log canonical, then clearly $(X, S + c\,\Delta)$ is plt for $c < 1$. Conversely, looking at the proof of Exercise 6.7, we see also that if $a(E, X, S + c\,\Delta)$ is greater than -1 for all positive c less than one, then as c approaches one, we can conclude that $a(E, X, S + \Delta) \geq -1$.

Similar arguments apply to the pair (S, Δ). □

SOLUTION TO 6.37.

We indicate only the key steps. We start with the equation $x_0^2 + y_0^{2d+1} = 0$. By successive blowups we get new coordinates and equations

$$x_1^2 + y_1^{2d-1} = 0, \quad x_2^2 + y_2^{2d-3} = 0, \; \ldots, \; x_d^2 + y_d = 0.$$

At the last step the curve became smooth, but it is tangent to the exceptional divisor. We need one more blowup. After that everything is smooth, and any two exceptional curves and the birational transform of C intersect transversely, but there are three of these curves meeting in one point. Thus we need one more blowup to get a log resolution. Using this resolution, we compute that the the log canonical threshold is $\frac{1}{2} + \frac{1}{2d+1}$. □

SOLUTION TO 6.43. We have already seen in the proof of Theorem 6.40 that the log canonical threshold is bounded above by $\frac{1}{i_0}$. Thus it suffices to show that the pair $(S, \frac{1}{i_0}C)$ is log canonical at P.

The hypothesis implies that f can be written in the form $f = x^{i_0} g$ where g is not divisible by x and its multiplicity at P is at most i_0. Thus in a neighborhood of P, we can write $C = i_0 L + D$ where L is the smooth divisor defined by the vanishing of the coordinate x and where D is some other divisor whose multiplicity at P is at most i_0. To check that $(S, \frac{1}{i_0}C) = (S, L + \frac{1}{i_0}D)$ is log canonical at P, then, we need only check that $(L, \frac{1}{i_0}D|_L)$ is log canonical at P by inversion of adjunction. Since the multiplicity of D is at most i_0 at P, this is immediate. □

Solution to 6.44.

Let $f_d(x, y)$ be weighted homogeneous of degree d and let α be a nonzero root of $f_d(x, 1)$. We claim that $x^b - \alpha^b y^a$ divides f_d. To see this, we can carry on the division algorithm until the highest x-power in the remainder is less than b. That is,

$$f_d(x, y) = g_{d-ab}(x, y)(x^b - \alpha^b y^a) + \gamma \cdot y^c h(x, y),$$

where y does not divide h and the highest x-power in h is $b' < b$. Thus the degree of h is $ab' < ab$. If $x^u y^v$ is any monomial in h then $au + bv = ab'$. Since a and b are relatively prime, this gives that b divides $b' - u < b$. Thus h is a power of x. Substituting $x = \alpha$, $y = 1$ now gives $\gamma = 0$. □

Solution to 6.45.

After any coordinate change, the equation looks like

$$g_m = q(x, y, z)^2 + (\text{higher terms})$$

where q is a quadratic form of rank three. It is easy to check that in this case, we are in one of the following three cases:

1. x^2, y^2, z^2 all appear in q;
2. x^2, yz all appear in q (up to permuting the coordinates);
3. xy, yz, zx all appear in q.

In the first case, $w(x)^4, w(y)^4, w(z)^4 \leq \operatorname{mult}_w g_m$, so

$$w(x) + w(y) + w(z) \leq \tfrac{3}{4} \operatorname{mult}_w g_m.$$

Similar simple inequalities take care of the other two cases. This shows that the best bound we can get trying to use a threefold analog of Theorem 6.40 is $\frac{3}{4}$.

In order to compute the log canonical threshold of D at the origin, we blow up to get $f : Y \to \mathbb{A}^3$. Let E be the exceptional divisor and Q the birational transform of the quadric cone $x^2 + y^2 + z^2 = 0$. Note that E and Q intersect transversally, and the key point is to understand what happens along $E \cap Q$. Here we locally have to compute with (Y, Δ) where $\Delta = c|4E + 2Q, mE| - 2E$. We can choose local coordinates such that E and Q are coordinate hyperplanes, so this is essentially a two-dimensional problem and already solved above. Details are left to the reader. □

Solution to 6.56.

Assume that $\mathcal{O}_X = f_*\mathcal{O}_Z(\sum k_i E_i)$. Pulling it back to Z we get an injection $f^*\mathcal{O}_X = \mathcal{O}_Z \hookrightarrow \mathcal{O}_Z(\sum k_i E_i)$ which shows that $k_i \geq 0$. Conversely, if every k_i is non-negative then we have an injection $\mathcal{O}_Z \hookrightarrow \mathcal{O}_Z(\sum k_i E_i)$ which can be pushed forward to $f_*\mathcal{O}_Z \hookrightarrow f_*\mathcal{O}_Z(\sum k_i E_i)$. Since X is normal, $f_*\mathcal{O}_Z = \mathcal{O}_X$ and the resulting map $\mathcal{O}_X \hookrightarrow f_*\mathcal{O}_Z(\sum k_i E_i)$ is an isomorphism outside the image of the exceptional set, hence outside a codimension at least two set $W \subset X$. Thus every local section of $f_*\mathcal{O}_Z(\sum k_i E_i)$ can be viewed as a rational function on X which is regular outside W. Since X is normal, this implies that such functions are regular everywhere, thus $f_*\mathcal{O}_Z(\sum k_i E_i) = \mathcal{O}_X$. $\square$

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