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MINIMAL MODELS OF RATIONAL SURFACES OVER ARBITRARY FIELDS

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Abstract. In this article all the types of minimal models of smooth rational surfaces defined over an arbitrary field are described.

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Introduction and statement of the main results

Let F be a complete smooth algebraic surface over a field k. F is said to be a (relatively) minimal model if any birational morphism $F \longrightarrow F'$ to a smooth surface F' is an isomorphism. As is customary, let

$$q(F) = \dim H^1(F, \mathcal{O}_F), \quad p(F) = \dim H^2(F, \mathcal{O}_F),$$
$$P_n(F) = \dim H^0(F, \Omega_F^{\otimes n}),$$

where $P_n(F)$ is the n-genus for $n \geqslant 1$, and Ω_F is the canonical invertible sheaf.

In this article we study minimal projective geometrically irreducible surfaces satisfying the conditions

$$q(F) = P_2(F) = 0. \tag{$*$}$$

According to Castelnuovo's rationality criterion (see for example [1], and also §2 of the present article), the conditions ($*$) characterize rational surfaces, i.e. those surfaces F over k such that over an algebraic closure $\bar{k}$ of k, $\bar{F} = F_{\bar{k}}$ becomes birationally equivalent to the projective plane $\mathbf{P}^2_{\bar{k}}$. It is well known (see [1], [7], [11] or [8]) that over an algebraically closed field the minimal models of rational surfaces are listed as the projective plane $\mathbf{P}^2_k$ and the series of scrolls $\mathbf{F}_N$ with $N \geqslant 0, N \neq 1$. For a nonclosed (but perfect) field k (see [11]), it is known only that one can bring any rational surface to one of the three Enriques-Manin standard types by birational transformations over k. In this article we show that in fact all minimal rational surfaces over any field k are themselves contained in two of the three families of standard Enriques-Manin forms (see Remark 2 on p. 27).

In §1 we prove the following result.

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THEOREM 1. *Any minimal surface F satisfying* (*) *is either isomorphic to* $\mathbf{P}_k^2$, *or to a quadric* $Q \subset \mathbf{P}_k^3$ *having* Pic $Q \simeq \mathbf{Z}$, *or belongs to one of the two following families of surfaces, defined by the conditions*:

I. Pic $F \simeq \mathbf{Z}$, *and it is generated by the anticanonical ample sheaf* Ω_F^{-1}.

II. Pic $F \simeq \mathbf{Z} \oplus \mathbf{Z}$, *and there exists a morphism* $f: F \longrightarrow C$ *having generic fiber* F_η *and the base* C *smooth curves of genus* 0. *For every closed point* $t \in C$ *the fiber* F_t *is irreducible over the residue field* $k(t)$, *is geometrically reduced, and has* $H^1(F_t, \mathcal{O}_{F_t}) = 0$. *Each nonsmooth geometrical fiber* F_t *is isomorphic to a pair of lines meeting in a point,* $\mathbf{P}_{\bar{k}}^1 \vee \mathbf{P}_{\bar{k}}^1$.

REMARK. A quadric $Q \subset \mathbf{P}_k^3$ having Pic $Q \simeq \mathbf{Z} \oplus \mathbf{Z}$ obviously belongs to the family II.

This theorem is the natural generalization of the corresponding result for an algebraically closed field k (see [11] and [7]), and its proof involves the usual technical tools: intersection theory, the Riemann-Roch theorem, and the classical adjunction lemma (Lemma 2).

In §2 we give a new proof of Castelnuovo's rationality criterion in characteristic $p > 0$ (Theorem 2). In contrast to the constructive and extremely lengthy proof of Zariski [15], [16] we use a technique of lifting to characteristic 0 to reduce the proof to the simple and conceptually transparent argument of Kodaira [7] in the case of the complex field $k = \mathbf{C}$. A different proof, using l-adic cohomology and the Brauer group, was given by M. Artin (see for example [8]).

In §3 we study the geometry of surface of families I and II. The surfaces of family I belong to the so-called del Pezzo surfaces (smooth surfaces having ample Ω_F^{-1}). The geometry of these has been well studied (see [10] and [11]). They have $1 \leqslant (\Omega_F \cdot \Omega_F) \leqslant 9$, and for $n = (\Omega_F \cdot \Omega_F) \geqslant 3$ they are surfaces of degree n in $\mathbf{P}_k^n$. At the start of §3 we give without proof a very short list of the basic properties of del Pezzo surfaces.

For surfaces in family II we prove the following theorem.

THEOREM 3. *Let* $f: F \longrightarrow C$ *be a morphism as in Theorem* 1, *where* C *is a smooth curve, not necessarily of genus* 0. *Then the following assertions hold*:

(1) *The sheaf* $f_*\Omega_F^{-1}$ *is locally free on* C *and of rank* 3; *the natural map* $\varphi: F \longrightarrow \mathbf{P}_C(f_*\Omega_F^{-1})$ *is an isomorphic embedding over* C, *mapping each fiber* F_t *into a conic of* $\mathbf{P}_{k(t)}^2$.

(2) *If* f *is a smooth morphism and* $p(C) = \dim H^1(C, \mathcal{O}_C) = 0$, *then either* $F \simeq C \times C'$, *where* C' *is a smooth curve of genus* 0 *not having any* k-*points, or* $F \simeq \mathbf{P}_C(E)$, *where* E *is some vector bundle over* C *of rank* 2. *In either case* $\bar{F} \simeq \mathbf{F}_N$ *with* $N \geqslant 0$, *where* $\mathbf{F}_N$ *is the standard scroll, and* $(\Omega_F \cdot \Omega_F) = 8$. *If* N *is odd, then* $C \simeq \mathbf{P}_k^1$.

(3) *If the morphism* f *is not smooth, then*

$$\operatorname{Pic} F = f^* \operatorname{Pic} C + \mathbf{Z} \cdot \Omega_F^{-1}.$$

Let $r = \Sigma_1^s \deg t_i$, *where* $\{t_1, \ldots, t_s\}$ *is the set of all points having degenerate fibers, and suppose* $p(C) = 0$; *then* rk Pic $\bar{F} = r + 2$ *and* $(\Omega_F \cdot \Omega_F) = 8 - r$. *If* $p(C) = 0$ *and* $(\Omega_F \cdot \Omega_F)$ *is odd, then* $C \simeq \mathbf{P}_k^1$.

(4) *The set of all surfaces in family* II *with a fixed smooth base curve* C (*not necessarily of genus* 0) *and birationally equivalent over* C *is in natural bijection with the set of all maximal orders over* C *in the quaternion algebra* A_η *corresponding to the generic fiber* F_η *over* $k(\eta)$.

We note that surfaces in family I are minimal, since Pic $F \simeq \mathbf{Z}$. It is also obvious that surfaces in family II are C-minimal, since the fibers of the morphism $F \longrightarrow C$ are irreducible.

THEOREM 4. *All the surfaces of family* II *are minimal, with the following exceptions*:

(a) $(\Omega_F \cdot \Omega_F) = 8$ *and* $\bar{F} \simeq \mathbf{F}_1$, *where* $\mathbf{F}_1$ *is the standard scroll–the image of* $\mathbf{P}^2_{\bar{k}}$ *under the blow-up of one point.*

(b) $(\Omega_F \cdot \Omega_F) = 3, 5,$ *or* 6.

THEOREM 5. *Suppose that F belongs to family* II.

(1) *If* $(\Omega_F \cdot \Omega_F) = 3, 5$ *or* 6, *then F is a del Pezzo surface.*

(2) *If* $(\Omega_F \cdot \Omega_F) = 8$ *and* $\bar{F} \not\simeq \mathbf{F}_N$ *with* $N \geqslant 2$, *i.e. if* $N = 0$ *or* 1, *then F is a del Pezzo surface.*

(3) *If* $(\Omega_F \cdot \Omega_F) = 1, 2$ *or* 4, *then F is a del Pezzo surface if and only if it has two distinct representations in the standard form* II.

There do not exist minimal smooth rational surfaces F with $(\Omega_F \cdot \Omega_F) = 7$.

In §4 we establish an analog of Theorem 1 for surfaces satisfying (*), on which a finite group G acts; that is, with a given representation $G \longrightarrow \mathrm{Aut}_k F$. For these surfaces one defines as usual the notion of G-map (and in particular G-morphism), G-minimal model, and so on.

As Manin showed in [11], the birational classification of rational G-surfaces is completely analogous to the birational classification of rational surfaces over fields that are not algebraically closed, and can be carried out by a unified approach using the adjunction lemma. In the case that k is algebraically closed, the birational classification of G-surfaces is equivalent to the problem of describing up to conjugacy the finite subgroups G in the Cremona group of $\mathbf{P}^2_k$. In [11] Manin describes the standard forms of G-surfaces up to birational G-equivalence, but only for Abelian groups G.

The following theorem strengthens and generalizes this result to arbitrary finite groups G.

Following [11], we let $P(F)$ denote the group generated by the classes of G-invariant divisors. Then it is easily seen that $\Omega_F \in P(F)$.

THEOREM 1G. *The assertions of Theorem* 1 *hold for any G-minimal surface F satisfying* (*), *provided that we replace ordinary morphisms by G-morphisms,* Pic F *by* $P(F)$, *and k-irreducibility by G-irreducibility.*

We remark that all the results of §3 can be stated and proved in an analogous manner for rational G-surfaces.

§1. The proof of Theorem 1

LEMMA 1. *Let F be a complete smooth surface over a field k, let* $L \in$ Pic F *be an invertible sheaf such that* $(L \cdot L) > 0$, $\dim H^0(F, L) \geqslant 2$ *and the complete linear system of curves* $|L|$ *(the zeros of sections of* $H^0(F, L)$*) has no fixed components. Then* $H^0(X, \mathcal{O}_X) = k$ *for any curve* $X \in |L|$.

PROOF. In the case that k is algebraically closed this is proved in [12]. The general case reduces to this by applying the simplest Künneth formulas (see [6], §6) to $F \otimes \bar{k}$, $L \otimes \bar{k}$ and $X \otimes \bar{k}$, where $\bar{k}$ is the algebraic closure of k.

LEMMA 2 (the adjunction lemma). *Let F be a smooth projective minimal surface over k, satisfying* (*); *then for any invertible sheaf $L \in \operatorname{Pic} F$ there exists an integer $n_L \geqslant 0$ such that $H^0(F, L \otimes \Omega_F^{\otimes n}) = 0$ for all $n > n_L$.*

This is proved exactly as in the case of an algebraically closed field (see, for example, [1], [7] or [8]).

LEMMA 3. *Let F be a surface as in Lemma 2, and suppose that* $\operatorname{Pic} F \neq \mathbf{Z} \cdot \Omega_F$. *Then F has an invertible sheaf L satisfying the following conditions*:

(1) $\dim H^0(F, L) \geqslant 1$ *and* $\dim H^0(F, L \otimes \Omega_F) = 0$.

(2) $(L \cdot \Omega_F) < 0$.

(3) *For any section $s \in H^0(F, L)$ the curve X of zeros of s is reduced and irreducible.*

PROOF. First of all note that if there exists a sheaf L' satisfying (1) and (2), then there also exists a sheaf L satisfying all three properties. For let $s' \in H^0(F, L')$ be some section, having divisor of zeros $X' = \Sigma_i r_i X_i'$ with $r_i > 0$, where the X_i' are reduced and irreducible components. Then any invertible sheaf $\mathcal{O}_F(X_i')$ satisfies (1), since

$$\dim H^0(F, \mathcal{O}_F(X_i') \otimes \Omega_F) \leqslant \dim H^0\left(F, \mathcal{O}_F\left(\sum_i r_i X_i'\right) \otimes \Omega_F\right) = 0$$

and obviously $\dim H^0(F, \mathcal{O}_F(X_i')) \geqslant 1$. It is then obvious that at least one of the sheaves $\mathcal{O}_F(X_i')$ satisfies (2), since $(\mathcal{O}_F(\Sigma_i r_i X_i') \cdot \Omega_F) < 0$.

If among the divisors of zeros of sections of $L_i' = \mathcal{O}_F(X_i')$ there remain reducible or multiple curves, we can repeat the process of separating off components; this process obviously terminates, since at each step the degree of L with respect to some fixed ample sheaf $\mathcal{O}_F(1)$ decreases.

We will prove the existence of a sheaf L having properties (1) and (2) using Lemma 2. Since $\operatorname{Pic} F \neq \mathbf{Z} \cdot \Omega_F$, we can find a very ample sheaf $H \in \operatorname{Pic} F$ with $H \notin \mathbf{Z} \cdot \Omega_F \subset \operatorname{Pic} F$. By Lemma 2 there exists an integer n_H such that the sheaf $L = H \otimes \Omega_F^{\otimes n_H}$ satisfies (1). If $(L \cdot \Omega_F) < 0$, everything is proved. If $(L \cdot \Omega_F) \geqslant 0$, then $(L \cdot L) < 0$ and $(\Omega_F \cdot \Omega_F) > 0$. Indeed, by the Riemann-Roch theorem,

$$0 = \dim H^0(F, L \otimes \Omega_F) \geqslant \frac{(L \cdot L \otimes \Omega_F)}{2} + 1.$$

Here $H^2(F, L \otimes \Omega_F) = H^0(F, L^{-1}) = 0$, since $L \not\cong \mathcal{O}_F$ (otherwise $H \in \mathbf{Z} \cdot \Omega_F$), and hence $(L \cdot L) + (L \cdot \Omega_F) \leqslant -2$, so that $(L \cdot L) \leqslant -2$. Furthermore, we have

$$0 < (H \cdot H) = (L \cdot L) - 2n_H (L \cdot \Omega_F) + n_H^2 (\Omega_F \cdot \Omega_F),$$

and hence $(\Omega_F \cdot \Omega_F) > 0$.

We will try to find the desired sheaf in the form $L_n = L^{-1} \otimes \Omega_F^{-n}$. For n sufficiently large we have $H^2(F, L^{-1} \otimes \Omega_F^{-n}) = 0$; for by Serre duality this is equivalent to $H^0(F, L \otimes \Omega_F^{n+1}) = 0$ for large n, which is true by Lemma 2. Hence for large n the Riemann-Roch theorem provides us with the inequality

$$\dim H^0(F, L^{-1} \otimes \Omega_F^{-n}) \geqslant \frac{(L \cdot L) + (2n+1)(L \cdot \Omega_F) + n^2 (\Omega_F \cdot \Omega_F)}{2} + 1.$$

Since $(\Omega_F \cdot \Omega_F) > 0$, the right-hand side is positive for large n. Thus we can find some minimal value $n_0 > 0$ such that

$$\dim H^0(F, L^{-1} \otimes \Omega_F^{-n_0}) \geqslant 1$$

and

$$\dim H^0(F, L^{-1} \otimes \Omega_F^{-n_0+1}) = 0,$$

since $H^0(F, L^{-1}) = 0$. Thus (1) holds for L_{n_0}; (2) also holds:

$$(L^{-1} \otimes \Omega_F^{-n_0} \cdot \Omega_F) = -(L \cdot \Omega_F) - n_0 (\Omega_F \cdot \Omega_F) < 0.$$

This proves Lemma 3.

LEMMA 4. *Under the hypotheses of Theorem* 1 *suppose that on F there is an invertible sheaf L satisfying properties* (1)–(3) *of Lemma* 3. *Then the following assertions hold*:

(a) $(L \cdot L) \geqslant 0$ *and* $\dim H^1(F, L) = 0$.

(b) *There exists an invertible sheaf* L' *satisfying* (1)–(3) *of Lemma* 3, *and for which* $(L' \cdot L') \leqslant 2$.

PROOF. (a) Suppose the contrary; that is, $(L \cdot L) < 0$. Then $\dim H^0(F, L) = 1$, since according to (3) the linear system $|L|$ cannot have fixed components. Let X be the (unique) reduced irreducible curve such that $L = \mathcal{O}_F(X)$. Consider the short exact sequence of sheaves

$$0 \to \mathcal{O}_F(-X) \to \mathcal{O}_F \to \mathcal{O}_X \to 0$$

and the corresponding long exact cohomology sequence

$$\ldots \to H^1(F, \mathcal{O}_F) \to H^1(X, \mathcal{O}_X) \to H^2(F, \mathcal{O}_F(-X)) \to \ldots. \tag{1.1}$$

Here $H^1(F, \mathcal{O}_F) = 0$ by hypothesis, $H^2(F, \mathcal{O}_F(-X)) = H^0(F, L \otimes \Omega_F) = 0$ by (1), and hence $H^1(X, \mathcal{O}_X) = 0$. On the other hand, we use the formula for the genus of a curve on a surface:

$$\dim H^1(X, \mathcal{O}_X) = \frac{(\mathcal{O}_F(X) \cdot \mathcal{O}_F(X)) + (\mathcal{O}_F(X) \cdot \Omega_F)}{2} + \dim H^0(X, \mathcal{O}_X). \tag{1.2}$$

Substituting $\dim H^1(X, \mathcal{O}_X) = 0$, and setting $\Omega_F = \mathcal{O}_F(K)$, with K the canonical divisor on F, we rewrite (1.2) in the form

$$(X \cdot X) + (X \cdot K) = -2\dim H^0(X, \mathcal{O}_X), \tag{1.3}$$

where for any divisors D_1 and D_2 the symbol $(D_1 \cdot D_2)$ denotes the intersection number, equal to $(\mathcal{O}_F(D_1) \cdot \mathcal{O}_F(D_2))$. Interpreting the intersection number as an Euler-Poincaré characteristic (as in [9]), and using the Künneth formula, we observe that $(D_1 \cdot D_2) = (\bar{D}_1 \cdot \bar{D}_2)$ on the surface $\bar{F}$ over $\bar{k}$. Hence it follows from (1.3) that

$$(\bar{X} \cdot \bar{X}) + (\bar{X} \cdot \bar{K}) = -2\dim_{\bar{k}} H^0(\bar{X}, \mathcal{O}_{\bar{X}}).$$

Let $\bar{X} = q \sum_1^m \bar{X}_i$ be the decomposition into irreducible components, where q is the multiplicity (if k is not a perfect field, then in general $q \neq 1$).

The Galois group $\mathrm{Gal}(\bar{k}_s/k)$ of the separable closure acts transitively on the set of components $\{\bar{X}_i\}$ and preserves the intersection numbers, so that $(\bar{X}_i \cdot \bar{X}) = (\bar{X}_j \cdot \bar{X})$ and $(\bar{X}_i \cdot \bar{K}) = (\bar{X}_j \cdot \bar{K})$ for any i and j. We have

$$qm(\bar{X}_i \cdot \bar{X}) + qm(\bar{X}_i \cdot \bar{K}) = -2\dim_{\bar{k}} H^0(\bar{X}, \mathcal{O}_X).$$

From the condition $(L \cdot \Omega_F) < 0$ we get $(\bar{X}_i \cdot \bar{K}) < 0$, and from the hypothesis $(L \cdot L) = (X \cdot X) < 0$ it follows that $(\bar{X}_i \cdot \bar{X}) < 0$. Furthermore, $(\bar{X}_i \cdot \bar{X}) = (\bar{X}_i \cdot q\Sigma\bar{X}_i) \leqslant -q$.

Since X is reduced and irreducible, $H^0(X, \mathcal{O}_X)$ is a field, which is a finite extension of k; the degree of the separable part of this extension is equal to the number of geometrical connected components of the curve $\bar{X}$; and the degree of the inseparable part is not greater than the geometrical multiplicity q. Hence $\dim H^0(X, \mathcal{O}_X) \leqslant mq$, and we get the inequality

$$qm(\bar{X}_i \cdot \bar{X}) + qm(\bar{X}_i \cdot \bar{K}) \geqslant -2qm,$$

which, because of the above-stated restrictions, has the unique solution $(\bar{X}_i \cdot \bar{X}) = (\bar{X}_i \cdot \bar{K}) = -1$ and $q = 1$. Thus each irreducible component $\bar{X}_i$ is a connected component. Hence $(\bar{X}_i \cdot \bar{X}_j) = 0$ for $i \neq j$, $(X_i \cdot X_i) = -1$ and $(\bar{X}_i \cdot \bar{K}) = -1$. But in view of the Castelnuovo contractibility criterion (see for example [18]), these conditions imply that X is an exceptional curve of the first kind on F. Since F is minimal, we have obtained a contradiction, which means that the supposition $(L \cdot L) < 0$ is false.

We now prove that $H^1(F, L) = 0$. Let X be any curve such that $L \simeq \mathcal{O}_F(X)$. We write the short exact sequence of sheaves

$$0 \to \mathcal{O}_F(-X) \otimes \Omega_F \to \Omega_F \to \mathcal{O}_X \otimes \Omega_F \to 0.$$

From the corresponding cohomology sequence we get

$$H^1(F, \mathcal{O}_F(-X) \otimes \Omega_F) = H^0(X, \mathcal{O}_X \otimes \Omega_F) = 0,$$

since by hypothesis $\deg_k(\mathcal{O}_X \otimes \Omega_F) = (L \cdot \Omega) < 0$ on X. Thus by Serre duality we get

$$\dim H^1(F, \mathcal{O}_F(-X) \otimes \Omega) = \dim H^1(F, \mathcal{O}(X)) = 0.$$

This proves (a).

(b) Note that if $(L \cdot L) > 0$ then

$$\dim H^0(F, L) = (L \cdot L) + 2. \tag{1.4}$$

For, in the Riemann-Roch formula, $H^1(F, L) = 0$ as just proved; and

$$\dim H^2(F, L) = \dim H^0(F, L^{-1} \otimes \Omega_F) = 0,$$

since $\dim H^0(F, \Omega_F) = 0$ and $L^{-1} \simeq \mathcal{O}_F(-X)$ is a sheaf of ideals. Hence

$$\dim H^0(F, L) = \frac{(L \cdot L) - (L \cdot \Omega_F)}{2} + 1 \geqslant 2,$$

so that L satisfies the conditions of Lemma 1.

Let X be any curve in the linear system $|L|$. Then by Lemma 1 $H^0(X, \mathcal{O}_X) = k$. On the other hand, as in the exact sequence (1.1), $H^1(X, \mathcal{O}_X) = 0$. Thus from the genus formula (1.2) we get

$$(L \cdot \Omega_F) = -(L \cdot L) - 2 \tag{1.5}$$

and (1.4) follows from the Riemann-Roch formula.

Now suppose that $(L \cdot L) \geqslant 3$. Consider the invertible sheaf $\Omega_F \otimes L^2$. Let us check that it satisfies (1) and (2) of Lemma 3. Since

$$\dim H^2(F, \Omega_F \otimes L^2) = \dim H^0(F, L^{-2}) = 0,$$

we can use (1.5) to obtain

$$\dim H^0(F, \Omega_F \otimes L^2) \geqslant \frac{4(L \cdot L) + 2(L \cdot \Omega_F)}{2} + 1 = (L \cdot L) - 1 \geqslant 2.$$

Furthermore, since

$$(L \cdot \Omega_F^2 \otimes L^2) = 2(L \cdot L) + 2(L \cdot \Omega_F) = -4$$

and the linear system $|L|$ is mobile according to (1.4), and has no fixed components by (3), we have

$$H^0(F, (\Omega_F \otimes L^2) \otimes \Omega_F) = 0.$$

Thus we have verified (1).

Before starting to check (2), we must recall Noether's formula:

$$\frac{(\Omega_F \cdot \Omega_F) + c_2(F)}{12} = \dim H^0(F, \mathcal{O}_F) - \dim H^1(F, \mathcal{O}_F) + \dim H^2(F, \mathcal{O}_F),$$

where $c_2(F)$ is the second Chern class of the tangent bundle to F. It is known (see [3]) that $c_2(F)$ can be expressed in terms of the l-adic Betti numbers b_i (where $l \neq \operatorname{char} k$):

$$c_2(F) = c_2(\overline{F}) = \sum_{i=0}^{4} (-1)^i b_i;$$

this is the topological Euler-Poincaré characteristic. For surfaces F satisfying (*) we have $b_0 = b_4 = 1$ and $b_1 = b_3 = 0$, and (1.6) takes the form

$$(\Omega_F \cdot \Omega_F) = 10 - b_2.$$

Since $b_2 \geqslant 1$ for a projective surface, we have $(\Omega_F \cdot \Omega_F) \leqslant 9$, as required.

Since $(L \cdot L) \geqslant 3$ and $(\Omega_F \cdot \Omega_F) \leqslant 9$, we have

$$(\Omega_F \otimes L^2 \cdot \Omega_F) = (\Omega_F \cdot \Omega_F) + 2(L \cdot \Omega_F) = (\Omega_F \cdot \Omega_F) - 2(L \cdot L) - 4 < 0.$$

Hence (2) also holds for the sheaf $\Omega_F \otimes L^2$. Furthermore,

$$H^1(F, \Omega_F \otimes L^2) = 0.$$

Indeed, the linear system $|L^2|$ has no fixed components, since $|L|$ has none. By Lemma 1 it follows that $H^0(Y, \mathcal{O}_Y) = k$ for any curve Y for which $L^2 = \mathcal{O}_F(Y)$. From the cohomology exact sequence associated to the short exact sequence of sheaves

$$0 \to \mathcal{O}_F(-Y) \to \mathcal{O}_F \to \mathcal{O}_Y \to 0,$$

we get immediately $H^1(F, L^{-2}) = H^1(F, \mathcal{O}_F(-Y)) = 0$, and from duality $H^1(F, \Omega_F \otimes L^2) = 0$.

Thus, using (1.5), the Riemann-Roch theorem gives us

$$\dim H^0(F, \Omega_F \otimes L^2) = (L \cdot L) - 1.$$

Starting from the sheaf $\Omega_F \otimes L^2$, we can now use the construction in the first part of the proof of Lemma 3 to find some invertible sheaf L_1 satisfying all of the three conditions in Lemma 3. By construction

$$\Omega_F \otimes L^2 = L_1 \otimes L_2,$$

where $H^0(F, L_2) \geqslant 1$. Using (1.4), we have

$$(L \cdot L) - 1 = \dim H^0(F, \Omega_F \otimes L^2) \geqslant \dim H^0(F, L_2) = (L_1 \cdot L_1) + 2,$$

and hence $(L_1 \cdot L_1) \leqslant (L \cdot L) - 3$. If $(L_1 \cdot L_1) \leqslant 2$ then L_1 is the required sheaf L'; if $(L_1 \cdot L_1) \geqslant 3$ then the same process may be repeated; this process obviously terminates. This proves (b), and with it Lemma 4.

LEMMA 5. *Under the conditions of Lemma* 4, *if* $(L_1 \cdot L_1) = 1$ *then* $F \simeq \mathbf{P}_k^2$; *if* $(L \cdot L) = 2$ *and there is no invertible sheaf* L' *on* F *with* $(L' \cdot L') = 0$ *and satisfying* (1)–(3) *of Lemma* 3, *then* $F \simeq Q$, *where* $Q \subset \mathbf{P}_k^3$ *is a smooth quadric with* Pic $Q \simeq \mathbf{Z}$.

PROOF. If $(L \cdot L) = 1$, then, according to (1.4), dim $H^0(F, L) = 3$. L is generated by its sections. For let $X \in |L|$ be any curve; then we have the exact sequence

$$0 \to H^0(F, \mathcal{O}_F) \to H^0(F, \mathcal{O}_F(X)) \to H^0(F, \mathcal{O}_X \otimes \mathcal{O}_F(X)) \to 0,$$

so that the trace of L on X has dimension 2, whereas if $|L|$ had a base point it could only be 1.

Thus L defines a morphism $\varphi_L\colon F \longrightarrow \mathbf{P}_k^2$ of degree 1. Since F is minimal, φ_L must be an isomorphism.

If $(L \cdot L) = 2$, then, just as in the case $(L \cdot L) = 1$, there is a morphism $\varphi_L\colon F \longrightarrow F' \subset \mathbf{P}_k^3$. The image $\varphi_L(F) = F'$ does not lie in any plane of $\mathbf{P}_k^3$, and cannot be a curve because $(L \cdot L) > 0$. Hence $F' = Q$ is a quadric of $\mathbf{P}_k^3$, and φ_L is a birational morphism. If Q is regular, then, because F is minimal, φ_L is an isomorphism and Q is smooth. There are two possible cases:

a) Pic $Q \simeq \mathbf{Z}$, with the plane section as a generator; here $\Omega_F \simeq \mathcal{O}_F(-2)$.

b) Pic $Q \simeq \mathbf{Z} \oplus \mathbf{Z}$, with generators the lines cut out by some tangent plane; here there is an L' with $(L' \cdot L') = 0$ provided by $\mathcal{O}_F(l)$, where l is either of these lines, and by the hypotheses of the lemma this case is excluded.

If Q is a quadratic cone and $F'' \longrightarrow Q$ is the resolution of its singular point, then φ_L': $F \longrightarrow F''$ is an isomorphism by the minimality of F. Let $Z \subset F''$ be the exceptional curve; one checks easily that $(Z \cdot Z) = -2$. We can also exclude this case, since $L' = L \otimes \mathcal{O}_F(-\varphi_L'^{-1}(Z))$ provides a sheaf with $(L' \cdot L') = 0$.

Thus there remains the unique possibility Pic $F \simeq \mathbf{Z}$, $F \simeq Q$, $\Omega_F \simeq \mathcal{O}_F(-2)$. This proves the lemma.

PROOF OF THEOREM 1. If Pic $F \simeq \mathbf{Z} \cdot \Omega_F^{-1}$, then F belongs to family I. In any other case, according to Lemmas 3 and 4, on F there is an invertible sheaf L with properties (1)–(3) of Lemma 3, and with $(L \cdot L) = 0$, 1 or 2. If $(L \cdot L) = 1$, then $F \simeq \mathbf{P}_k^2$ by Lemma 5. If the minimal selfintersection number $(L \cdot L)$ (taken over all L) is 2, then according to Lemma 5 F is a quadric in $\mathbf{P}_k^3$ with Pic $F \simeq \mathbf{Z}$. It remains to deal with the case $(L \cdot L) = 0$. Let us show that in this case F belongs to family II. From the Riemann-Roch theorem, using the second assertion in (a) of Lemma 4, we get

$$\dim H^0(F, L) = -\frac{(L \cdot \Omega_F)}{2} + 1 \geqslant 2,$$

since $(L \cdot \Omega_F) < 0$. The linear system $|L|$ is without fixed components (by (3) of Lemma 3), and $(L \cdot L) = 0$; hence L defines a morphism of F onto some curve. From the results of [17] it follows that the graded ring $R = \bigoplus_{m \geqslant 0} H^0(F, L^m)$ is finitely generated, and there exists a surjective morphism

$$f: F \to \operatorname{Proj} R = C,$$

where C is a complete nonsingular curve. In view of the Künneth formulas, f commutes with extensions of the base field, and we obtain

$$\bar{f}: \bar{F} \to \bar{C} = \operatorname{Proj}(R \otimes \bar{k}).$$

As shown in [17], $\bar{C}$ is a normal irreducible curve, and hence C is smooth over k; the function field $\bar{k}(\bar{C})$ is algebraically closed in $\bar{k}(\bar{F})$, and hence each fiber $\bar{F}_{\bar{t}}$ is connected, and the generic fiber F_η is geometrically irreducible. It follows that $\bar{f}_* \mathcal{O}_{\bar{F}} = \mathcal{O}_{\bar{C}}$ and $f_* \mathcal{O}_F = \mathcal{O}_C$. The remaining properties of f are contained in the following lemma.

LEMMA 6. *For any point $t \in C$, $H^1(F_t, \mathcal{O}_{F_t}) = 0$, the fiber F_t is irreducible over $k(t)$ and geometrically reduced. Every degenerate fiber $\bar{F}_{\bar{t}}$ (for $\bar{t} \in \bar{C}$) is of the form $\bar{F}_{\bar{t}} = \bar{X}_1 + \bar{X}_2$, where $\bar{X}_1$ and $\bar{X}_2$ are irreducible smooth curves of genus 0 satisfying $(\bar{X}_i \cdot \bar{X}_i) = -1$ and $(\bar{X}_1 \cdot \bar{X}_2) = 1$.*

PROOF. The morphism $f: F \to C$ is flat ([9], Lecture 6), and $R^q f_* \mathcal{O}_F = 0$ for $q \geqslant 2$ from considerations of dimension. Hence (see [9], Lecture 7) the sheaf $R^1 f_* \mathcal{O}_F$ is locally free over C, and

$$(R^1 f_* \mathcal{O}_F) \otimes k(t) = H^1(F_t, \mathcal{O}_{F_t}) \quad \forall t \in C.$$

But for some closed point $t \in C$ we have $F_t = X \in |L|$. From the exact sequence (1.1) we get $H^1(X, \mathcal{O}_X) = 0$, which proves the first assertion of the lemma.

Let $F_t = Y_1 + \cdots + Y_r$, with $r \geqslant 2$, and with the Y_i being irreducible curves on F. Since $F_t \in |L^m|$ for some $m \geqslant 1$,

$$(\mathcal{O}_F(F_t) \cdot \Omega_F) = m(L \cdot \Omega_F) < 0.$$

Without loss of generality we can assume that $(\mathcal{O}_F(Y_1) \cdot \Omega_F) < 0$. Clearly $(Y_1 \cdot F_t) = 0$, and since F_t is connected, $(Y_1 \cdot F_t - Y_1) > 0$. Hence $(Y_1 \cdot Y_1) < 0$. Since

$$(\mathcal{O}_F(Y_1) \otimes \Omega_F \cdot L) = (\Omega_F \cdot L) < 0,$$

we have

$$\dim H^0(F, \mathcal{O}_F(Y_1) \otimes \Omega) = 0.$$

Thus on F there is an invertible sheaf $\mathcal{O}_F(Y_1)$ satisfying conditions (1) and (2) of Lemma 3, and with $(\mathcal{O}_F(Y_1) \cdot \mathcal{O}_F(Y_1)) < 0$. From the first step in the proof of Lemma 3 it is clear that in this case F has a sheaf $\mathcal{O}_F(Y)$ satisfying all three conditions of Lemma 3, and with $(\mathcal{O}_F(Y) \cdot \mathcal{O}_F(Y)) < 0$. But this contradicts Lemma 4. This proves the irreducibility of the fibers.

Now let $\bar{F}_{\bar{t}}$ be some closed geometric fiber, with $\bar{t} \in \bar{C}$. Since $R^1 f_* \mathcal{O}_F = 0$, we have

$$H^0(\bar{F}_{\bar{t}}, \mathcal{O}_{\bar{F}_{\bar{t}}}) = \bar{f}_* \mathcal{O}_{\bar{F}} \otimes k(\bar{t}) = \mathcal{O}_{\bar{C}} \otimes k(\bar{t}) = \bar{k}.$$

Since $(\bar{F}_t \cdot \bar{F}_t) = 0$ and $H^1(\bar{F}_{\bar{t}}, \mathcal{O}_{\bar{F}_{\bar{t}}}) = 0$, the genus formula (1.3) gives $(\bar{F}_{\bar{t}} \cdot K_{\bar{F}}) = -2$. If $\bar{F}_{\bar{t}} = q\bar{X}_0$, where $\bar{X}_0$ is a reduced curve and q is a multiplicity (since the fiber F_t is irreducible its geometric irreducible components have the same multiplicity), we have similarly $(\bar{X}_0 \cdot K_{\bar{F}}) = -2$. But $(\bar{F}_{\bar{t}} \cdot K_{\bar{F}}) = q(\bar{X}_0 \cdot K_{\bar{F}})$; hence $q = 1$ and $\bar{F}_{\bar{t}}$ is reduced. Since every geometric fiber is reduced it follows that the generic fiber F_η is also geometrically reduced.

For the last assertion we note that in view of the irreducibility of F_t, the intersection number $(\bar{X}_i \cdot K_{\bar{F}})$, for any irreducible component $\bar{X}_i$ of the curve $F_t \otimes \bar{k}$, is independent of i. Hence from $(\bar{F}_{\bar{t}} \cdot K_{\bar{F}}) = -2$ it follows that $\bar{F}_{\bar{t}}$ can consist of at most 2 components. Let $F_{\bar{t}} = \bar{X}_1 + \bar{X}_2$; then $(\bar{X}_i \cdot K_{\bar{F}}) = -1$, and there are no possibilities other than $(\bar{X}_i \cdot \bar{X}_i) = -1$, $(\bar{X}_1 \cdot \bar{X}_2) = 1$ and $p(\bar{X}_i) = 0$. The lemma is proved.

COROLLARY. *For every degenerate fiber F_t the residue field $k(t)$ is separable over k.*

For let q be the degree of inseparability of $k(t)/k$. F_t decomposes into 2 components over some separable extension of k, and we can therefore assume that k is separably closed. Then $\bar{F}_t = q(\bar{X}_1 + \bar{X}_2)$. From the genus formula we have

$$(\bar{F}_{\bar{t}} \cdot \bar{F}_{\bar{t}}) + (\bar{F}_t \cdot K_{\bar{F}}) = -2\dim H^0(F_t, \mathcal{O}_{F_t}) = -2\,[k(t) : k] = -2q$$

and similarly

$$(\bar{X}_i \cdot \bar{X}_i) + q(\bar{X}_i \cdot K_{\bar{F}}) = -2q.$$

Since $(\bar{X}_i \cdot \bar{X}_i) = -1$, it follows that $q = 1$.

Let us conclude the proof of Theorem 1. From the Leray spectral sequence for the morphism f and the sheaves $\mathcal{O}_F$ and $G_{m,F}$ (where $G_{m,F}$ is the sheaf of invertible sections of $\mathcal{O}_F$) in the Zariski topology,

$$E_2^{p,q} = H^p(C, R^q f_* \mathcal{O}_F) \Rightarrow H^{p+q}(F, \mathcal{O}_F),$$
$$E_2^{p,q} = H^p(C, R^q f_* G_{m,F}) \Rightarrow H^{p+q}(F, G_{m,F}),$$

we obtain exact sequences for the initial terms:

$$0 \to H^1(C, f_* \mathcal{O}_F) \to H^1(F, \mathcal{O}_F) \to \dots,$$
$$0 \to H^1(C, f_* G_{m,F}) \to H^1(F, G_{m,F}) \to H^0(C, R^1 f_* G_{m,F}) \to \dots.$$

Since $f_* \mathcal{O}_F = \mathcal{O}_C$, we have that $f_* G_{m,F} = G_{m,C}$. Hence from the first exact sequence we get $H^1(C, \mathcal{O}_C) = 0$, and from the second (since $H^1(X, G_{m,F}) = \operatorname{Pic} X$ for any scheme X) we get the exact sequence

$$0 \to \operatorname{Pic} C \xrightarrow{f^*} \operatorname{Pic} F \to \operatorname{Pic} F/C \to \dots,$$

where Pic F/C is the group of global sections of the sheaf of relative Picard groups: this group is generated by irreducible components of fibers and the Picard group of the generic fiber F_η of f. In our case, by Lemma 6, all the fibers are irreducible, so that there are no nontrivial components of fibers. Furthermore, as was established at the beginning of the proof, F_η is a geometrically irreducible curve, and by Lemma 6 F_η is a smooth curve of genus 0. Hence Pic $F_\eta \simeq \mathbf{Z}$, and we also have Pic $C \simeq \mathbf{Z}$. $L \in$ Pic F is effective, but not ample (since $(L \cdot L) = 0$), so that Pic $F \not\simeq \mathbf{Z}$. Hence from (1.7) we get Pic $F \simeq \mathbf{Z} \oplus \mathbf{Z}$. The theorem is proved.

COROLLARY (ENRIQUES-MANIN). *Every smooth complete rational surface F' over k is birationally equivalent over k to one of the surfaces F in Theorem* 1.

In fact, Theorem 1 even implies that there is a birational k-morphism $F' \longrightarrow F$.

REMARKS. 1. If k is a perfect field, then it is known that every complete regular surface is smooth. Thus by the theorem on the resolution of singularities every reduced irreducible surface over k (in particular, a rational surface) can be birationally transformed into a complete smooth surface. It would be interesting to describe the complete and regular (but not smooth) minimal rational surfaces.

2. In addition to the surfaces (the standard forms) described in Theorem 1, there is another family of standard forms given in [11]—the degenerate del Pezzo surfaces, as defined in [11]. This class includes the complete smooth surfaces F over k obtained by resolving the singularities of normal rational surfaces F^* (having only ordinary double points), satisfying Pic $F^* \simeq \mathbf{Z} \cdot \Omega_{F^*}^{-1}$. It follows from our Theorem 1 that surfaces of this family either are nonminimal, or belong to family II. For example, if $F^* \subset \mathbf{P}_k^3$ is a cubic surface with 3 conjugate geometric double points, and $F \longrightarrow F^*$ is a resolution, then F contains a geometrically reducible exceptional curve of the first kind Z: the proper transform of the plane section of F^* passing through the 3 singular points. On contracting Z we obtain a del Pezzo surface of degree 6.

§2. The Castelnuovo rationality criterion

The following proposition is a partial converse of Theorem 1.

PROPOSITION 1 (a). *Every complete smooth surface F over k having ample sheaf Ω_F^{-1} satisfies* (*).

(b) (NOETHER'S LEMMA). *Let F be a smooth complete surface over k and let g: $F \longrightarrow C$ be a rational map onto a curve C of genus* 0, *whose generic fiber is a geometrically irreducible, geometrically reduced rational curve. Then F is a rational surface* (*and hence satisfies* (*)).

PROOF. a) Since Ω_F^{-1} is ample, it follows at once that

$$H^0(F, \Omega_F^m) = 0 \quad \forall m \geq 1,$$

and, in particular, $p = P_2 = 0$.

Since the conditions (*) are invariant under extension of the constant field, we can assume that k is algebraically closed. By Noether's formula (1.6), $(\Omega_F \cdot \Omega_F) + c_2(F) = 12(1 - q)$. Since $p(F) = 0$, the Picard scheme Pic$^0 F$ is reduced (see [9], Lecture 27), and

$$b_1(F, l) = \dim \operatorname{Pic}^0 F = q, \quad l \neq \operatorname{char} k.$$

This follows at once from the exact sequence

$$0 \to H^1(F, \mu_{l^n}) \to \operatorname{Pic} F \xrightarrow{l^n} \operatorname{Pic} F \to \ldots,$$

corresponding to the Kummer short exact sequence

$$0 \to \mu_{l^n} \to G_{m,F} \xrightarrow{l^n} G_{m,F} \to 0.$$

Substituting in the Noether formula, we get

$$(\Omega_F \cdot \Omega_F) = 8(1-q) + 2 - b_2(F, l).$$

Since $b_2(F, l) \geqslant 1$, if $q > 1$ we would get $(\Omega_F \cdot \Omega_F) < 0$, contradicting the ampleness of Ω_F^{-1}. If $q = 1$ then $b_2(F, l) = 1$, but in this case the Albanese map $F \longrightarrow A$ is a morphism onto an elliptic curve ($\dim A = q = 1$), and the fiber F_a satisfies $(F_a \cdot F_a) = 0$, contradicting the inequalities

$$1 \leqslant \operatorname{rk} \operatorname{Pic} F/\operatorname{Pic}^0 F \leqslant b_2(F, l) = 1$$

and the ampleness of Ω_F^{-1}.

b) Let G be the complete curve of genus 0 over $k(C)$ which is birationally equivalent to F_η. Then G is a smooth curve, and its anticanonical map embeds it isomorphically as a nondegenerate conic $Q \subset \mathbf{P}^2_{k(C)}$. If $\bar{k}$ is the algebraic closure of k, then by Tsen's theorem the conic $Q \otimes \bar{k}(C)$ has a $\bar{k}(C)$-point, and is hence isomorphic to $\mathbf{P}^1_{\bar{k}(C)}$. This means that $F \otimes \bar{k}$ is birationally equivalent over $\bar{k}$ to the surface $\mathbf{P}^1_{\bar{k}} \times C \otimes \bar{k}$. Since by hypothesis $C \otimes \bar{k} \simeq \mathbf{P}^1_{\bar{k}}$, $\bar{F}$ is a rational surface, and because conditions (*) are birationally invariant, they hold for $\bar{F}$. This completes the proof.

In the case of an algebraically closed field k the conditions (*) for F make up the famous Castelnuovo rationality criterion. The most transparent proof of this criterion in characteristic 0 was given by Kodaira [7]. In characteristic $p > 0$ this was proved by Zariski [15], [16]. Artin has extended Kodaira's proof to characteristic $p > 0$, replacing the classical cohomology by l-adic cohomology (see for example [8]).

Our proof is a reduction to Kodaira's proof by means of a lifting of F to characteristic 0.

THEOREM 2 (the Castelnuovo rationality criterion). *A smooth projective surface F over an algebraically closed field k is rational if and only if $q(F) = P_2(F) = 0$.*

PROOF. We only have to prove that (*) implies rationality. We can assume that F is minimal. Since k is algebraically closed, according to Noether's lemma (see (b) of Proposition 1) any surface of family II is rational; a quadric of $\mathbf{P}^3_k$ is also rational. Hence it remains to consider the case that F belongs to family I.

The following lemma completes the proof of Theorem 2.

LEMMA 7. *Over an algebraically closed field k there do not exist any smooth surfaces F in family* I.

PROOF. If char $k = 0$ this was proved by Kodaira [7]; his argument is very simple. We can assume that $k = \mathbf{C}$; then the exponential exact sequence gives an isomorphism

$$H^2(F, \mathbf{Z}) \simeq \operatorname{Pic} F.$$

But by hypothesis $\operatorname{Pic} F \simeq \mathbf{Z} \cdot \Omega_F^{-1}$. Then by Poincaré duality we should have $(\Omega_F \cdot \Omega_F) = \pm 1$, whereas Noether's formula gives $(\Omega_F \cdot \Omega_F) = 9$; this is a contradiction.

Now suppose that char $k = p > 0$. Since $\dim H^0(F, \Omega_F^{-1}) \geqslant (\Omega_F \cdot \Omega_F) + 1$, there is a curve $X \in |\Omega_F^{-1}|$. From the condition $\operatorname{Pic} F \simeq \mathbf{Z} \cdot \Omega_F^{-1}$ it follows that X is reduced and irreducible, and from the genus formula (1.2) we have $p(X) = 1$.

Let us show that F satisfies the sufficient condition for lifting to characteristic 0; that is, $H^2(F, T_F) = 0$, where T_F is the sheaf of germs of sections of the tangent bundle.

We consider the sheaf exact sequence

$$0 \to \mathcal{O}_F((n-1)X) \otimes T_F \to \mathcal{O}_F(nX) \otimes T_F \to \mathcal{O}_X \otimes \mathcal{O}_F(nX) \otimes T_F \to 0, \qquad n \geqslant 0,$$

and the corresponding cohomology sequence

$$\ldots \to H^1(X, \mathcal{O}_X \otimes \mathcal{O}_F(nX) \otimes T_F) \to H^2(F, \mathcal{O}_F((n-1)X) \otimes T_F) \to H^2(F, \mathcal{O}_F(nX) \otimes T_F) \to 0. \tag{2.1}$$

By Serre's theorem $H^2(F, \mathcal{O}_F(nX) \otimes T_F) = 0$ for sufficiently large n, since $\mathcal{O}_F(X) \simeq \Omega_F^{-1}$ is ample. Hence for $H^2(F, T_F)$ to vanish it is sufficient that

$$H^1(X, \mathcal{O}_X \otimes \mathcal{O}_F(nX) \otimes T_F) = 0 \quad \text{for all} \quad n \geqslant 1.$$

We consider two cases separately:

a) the linear system $|\Omega_F^{-1}|$ contains a smooth curve X;

b) all curves of $|\Omega_F^{-1}|$ are singular.

In case a) suppose that X is a smooth curve; then $T_X \simeq \mathcal{O}_X$, since $p(X) = 1$, and we have the sheaf exact sequence on X:

$$0 \to \mathcal{O}_X \to T_{F/X} \to N \to 0,$$

where $N = \mathcal{O}_X \otimes \mathcal{O}_F(X)$ is the normal sheaf to X in F. Taking the tensor product with $\mathcal{O}_F(nX)$ and passing to the cohomology sequence, we get

$$\ldots \to H^1(X, \mathcal{O}_X \otimes \mathcal{O}_F(nX)) \to H^1(X, \mathcal{O}_F(nX) \otimes T_{F/X}) \to H^1(X, \mathcal{O}_F(nX) \otimes N) \to 0. \tag{2.2}$$

Here

$$H^1(X, \mathcal{O}_X \otimes \mathcal{O}_F(nX)) = H^1(X, \mathcal{O}_F(nX) \otimes N) = 0$$

for $n \geqslant 1$ by the Riemann-Roch theorem on X, and hence

$$H^1(X, \mathcal{O}_F(nX) \otimes T_{F/X}) = 0;$$

thus $H^2(F, T_F) = 0$.

Case b) cannot occur. Indeed, according to [14], it is only possible for all curves of $|\Omega_F^{-1}|$ to be singular if char $k = 2$ or 3, and every curve X must have only a single cusp singularity (that is, a point like $t_1^2 - t_2^3 = 0$). Let us choose some linear pencil $M \subset |\Omega_F^{-1}|$. Then the geometrical locus of singular points of the elements of M is a certain curve Y. As remarked in [19], Y cannot be singular. The pencil M defines a purely inseparable cover $Y \longrightarrow \mathbf{P}^1$. It follows that Y is a rational curve, and, since it is nonsingular, that $p_a(Y) = 0$. On the other hand, since $\operatorname{Pic} F \simeq \mathbf{Z} \cdot \mathcal{O}_F(X)$, there exists some integer m such that $Y \sim mX$, and then $p_a(Y) = m(m-1)/2 + 1 > 0$; this is a contradiction.

Thus, by Grothendieck's theory (see SGA3 and [6], Chapter III), F can be lifted to characteristic 0, and, furthermore, the lifting can be algebraized, since F is projective. More precisely, there is a complete discrete valuation ring W with residue class field k and field of fractions K, with char $K = 0$, and there exists a flat proper morphism $h: \widetilde{F} \longrightarrow \operatorname{Spec} W$ having the closed fiber $\widetilde{F} \otimes k \simeq F$. Since the closed fiber is smooth, so is the generic fiber $\widetilde{F}_w$, where $w \in \operatorname{Spec} W$ is the generic point. Hence h is a smooth morphism. We have $h_* \mathcal{O}_{\widetilde{F}} = W$ and $R^i h_* \mathcal{O}_{\widetilde{F}} = 0$ for $i > 2$, since the fibers are 2-dimensional. Hence (see [9], Lecture 7) $R^2 h_* \mathcal{O}_{\widetilde{F}}$ is locally free, and since it is zero on the closed fiber, it is everywhere zero. Now, similarly, we have $R^1 h_* \mathcal{O}_{\widetilde{F}} = 0$. Thus for any fiber, and in particular for a geometric generic fiber,

$$p(\widetilde{F}_w \otimes \overline{K}) = q(\widetilde{F}_w \otimes \overline{K}) = 0.$$

The group Pic F, with its positive generator Ω_F^{-1}, lifts to $\widetilde{F}$, and because of the semicontinuity of the rank (see [13], Exposé X, (7.16.2)),

$$\operatorname{Pic}(\widetilde{F}_w \otimes \overline{K}) \simeq \mathbf{Z} \cdot \Omega^{-1}_{\widetilde{F} \otimes \overline{K}}$$

(the rank for the geometric generic fiber cannot be greater than that of the closed fiber; but rk Pic $F = 1$).

Now $\widetilde{F}_w \otimes \overline{K}$ is a surface over the algebraically closed field $\overline{K}$ of characteristic 0, and it satisfies all the properties of the surfaces of family I, which, as we have shown, is empty. Hence there also do not exist surfaces of family I over an algebraically closed field of characteristic $p > 0$. This proves the lemma, and with it Theorem 2.

§3. The geometry of the standard forms

DEFINITION. A smooth complete surface F over a field k is a *del Pezzo surface* if Ω_F^{-1} is an ample sheaf.

All the surfaces of family I of Theorem 1 are del Pezzo surfaces; some of the surfaces F of family II can also be del Pezzo surfaces.

We note the main properties of del Pezzo surfaces (see [10] and [11]).

(a) $1 \leqslant (\Omega_F \cdot \Omega_F) \leqslant 9$; this follows from (1.6).

(b) If $n = (\Omega_F \cdot \Omega_F) \geqslant 3$ then Ω_F^{-1} is very ample, and defines an isomorphic embedding of F in $\mathbf{P}_k^n$ as a surface of degree n; conversely every smooth (normal) surface F of degree $3 \leqslant n \leqslant 9$ in $\mathbf{P}_k^n$ and not lying in any hyperplane is a del Pezzo surface. If $n = 2$, then Ω_F^{-1} defines a double cover $F \longrightarrow \mathbf{P}_k^2$ with ramification a curve of degree 4. If $n = 1$, then Ω_F^{-2} defines a double cover $F \longrightarrow Q \subset \mathbf{P}_k^3$, with Q a quadratic cone, and the ramification curve is cut out on Q by some cubic not passing through the vertex.

(c) Every geometrically irreducible curve X on F having negative selfintersection number is an exceptional curve of the first kind. The number of such curves on $F \otimes \bar{k}$ is finite, and this number and the configuration of the curves has been determined (see [10], Chapter V, Theorem 4.3). If $\bar{F}$ does not contain exceptional curves, then either $n = 9$ and $\bar{F} \simeq \mathbf{P}_{\bar{k}}^2$, or $n = 8$ and $\bar{F} \simeq \mathbf{P}_{\bar{k}}^1 \times \mathbf{P}_{\bar{k}}^1$.

(d) Over an algebraically closed field k a del Pezzo surface F with $n = 9$ is isomorphic to $\mathbf{P}_k^2$; if $n = 8$ then either $F \simeq \mathbf{P}_k^1 \times \mathbf{P}_k^1$ or $F \simeq \mathbf{F}_1$ is the image of $\mathbf{P}_k^2$ under the blow-up with center in 1 point. If $1 \leqslant n \leqslant 7$ then F is a del Pezzo surface if and only if it is isomorphic to the image of $\mathbf{P}_k^2$ under the blow-up of $9 - n$ closed points, of which no 3 lie on a line, no 6 on a conic, and for $n = 1$ all eight do not lie on a cubic which is singular at one of them.

(e) Let $F \longrightarrow F'$ be a birational morphism of smooth projective surfaces. If F is a del Pezzo surface, so is F'. If F' is a del Pezzo surface, $(\Omega_F \cdot \Omega_F) \geq 1$, and all curves on $\bar{F}$ with negative intersection number are exceptional, then F is also a del Pezzo surface.

REMARK. In Manin's book [10] (Chapter IV, §§2.5, 2.8 and 4.3) the question of the sufficiency of the conditions of "general position" given in (d) is left open. The following simple argument answers this question.

Let $g: F \longrightarrow \mathbf{P}_k^2$ be the blow-up with center in the $r = 9 - n$ distinct points $x_1, \ldots, x_r$ of $\mathbf{P}_k^2$ (k is algebraically closed). g maps curves of the linear system $|\Omega_F^{-1}|$ into cubic curves through $x_1, \ldots, x_r$ (since $f_*\Omega_F = \Omega_{\mathbf{P}_k^2}$). Since $n \leq 8$, we have $(\Omega_F \cdot \Omega_F) \geq 1$ and

$$\dim H^0(F, \Omega_F^{-1}) = (\Omega_F \cdot \Omega_F) + 1 \geq 2.$$

By the numerical criterion for ampleness, Ω_F^{-1} will be ample provided that

(α) $|\Omega_F^{-1}|$ has no fixed components, and

(β) F contains no reduced irreducible curves Z with $(\mathcal{O}_F(Z) \cdot \Omega_F) = 0$; if $|\Omega_F^{-1}|$ is without fixed components then the morphism corresponding to Ω_F^{-m} contracts these curves into singular points.

In case (α), let Y be a fixed component; then $g(Y)$ is a component of all the cubics passing through $x_1, \ldots, x_r$, and hence $g(Y)$ is either a line, or a conic. If $g(Y)$ is a line passing through only 2 points x_i then the space of residual conics, passing through at least $r - 2$ points, has dimension strictly less that $(\Omega_F \cdot \Omega_F)$, the dimension of the linear system $|\Omega_F^{-1}|$. Hence $g(Y)$ passes through $s \geq 3$ points. The case that $g(Y)$ is a conic is similar.

In case (β) we automatically get $(Z \cdot Z) = -2$ from the negative definiteness of the intersection form on contracted curves and from the formula for the genus. Let $g(Z) \subset \mathbf{P}_k^2$ be a curve of degree m passing through x_i with multiplicity ν_i. Then

$$(Z \cdot Z) = m^2 - \sum_{i=1}^{r} \nu_i^2 = -2,$$

$$(\mathcal{O}_F(Z) \cdot \Omega_F) = 3m - \sum_{i=1}^{r} \nu_i = 0$$

or

$$\sum_{i=1}^{r} \nu_i^2 = m^2 + 2,$$

$$\sum_{i=1}^{r} \nu_i = 3m.$$

The quadratic form $\Sigma_1^r \nu_i^2$ takes on the minimum value $9m^2/r$ at $\nu_1 = \cdots = \nu_r = \nu$, with $\nu = 3m/r$; hence $9m^2/r \leq m^2 + 2$, or $(9 - r)m^2 \leq 2r$. It is clear from this that only the following possibilities can occur: $3 \leq r < 6$ and $m = 1$; $6 \leq r < 8$ and $m = 1$ or 2; $r = 8$ and $m \leq 3$ ($m \neq 4$ since $\nu = 12/8$ is not an integer, and the minimum is not attained). In the first two cases the curve $g(Z)$ can only be a line or a conic. From $(Z \cdot Z) = -2$ it follows that a line has to pass through 3 points x_i, and a conic through 6 points. In the case

$r = 8$, $m = 3$ one of the points must have multiplicity 2, and the others 1. Thus we have proved that Ω_F^{-1} is ample if and only if the conditions stated in the second sentence of (d) hold.

In the rest of this section we will mainly consider surfaces of family II.

PROOF OF THEOREM 3. (1) Since $p(F_t) = 0$, simple computations show that

$$H^1(F_t, \Omega_F^{-1} \otimes \mathcal{O}_{F_t}) = 0$$

for every fiber F_t, $t \in C$. Hence $R^1 f_* \Omega_F^{-1} = 0$, and the sheaf $f_* \Omega_F^{-1}$ is locally free of rank $\dim H^0(F_t, \Omega_F^{-1} \otimes \mathcal{O}_{F_t}) = 3$ (see [9], Lecture 7). Further, from the structure of the geometric fibers $\bar{F}_{\bar{t}}$ it is clear that the restriction of Ω_F^{-1} to every component of $\bar{F}_{\bar{t}}$ is a very ample sheaf. Hence we get the required isomorphic embedding over C

$$\varphi : F \to \mathbf{P}_C(f_* \Omega_F^{-1}).$$

Since

$$\deg_{\bar{k}} (\Omega_{\bar{F}}^{-1} \otimes \mathcal{O}_{\bar{F}_{\bar{t}}}) = (\Omega_{\bar{F}}^{-1} \cdot \mathcal{O}_{\bar{F}}(\bar{F}_{\bar{t}})) = 2,$$

$\varphi(\bar{F}_{\bar{t}})$ is a conic in the corresponding plane

$$\mathbf{P}(H^0(\bar{F}_{\bar{t}}, \Omega_{\bar{F}} \otimes \mathcal{O}_{\bar{F}_{\bar{t}}}) \simeq \mathbf{P}^2_{\bar{k}}.$$

If $\bar{F}_{\bar{t}}$ is reducible, then $\varphi(\bar{F}_{\bar{t}})$ is a pair of lines of $\mathbf{P}^2_{\bar{k}}$.

(2) Suppose that $p(C) = 0$ and that f is smooth. First of all we assume that there exists a section $s: C \longrightarrow F$, and set $s(C) = S$. Then, as in (1), $R^i f_* \mathcal{O}_F(S) = 0$ for $i \geqslant 1$, and $f_* \mathcal{O}_F(S)$ is a locally free sheaf of rank $\dim H^0(F_t, \mathcal{O}_F(S) \otimes \mathcal{O}_{F_t}) = 2$. Let E be the corresponding rank 2 vector bundle; then it is not difficult to check that the map $\psi: F \longrightarrow \mathbf{P}_C(E)$ is an isomorphism over C. From the classification of vector bundles over the line it is known that $\mathbf{P}_{\bar{C}}(E \otimes \bar{k}) \simeq \mathbf{F}_N$, where

$$\mathbf{F}_N = \mathrm{Proj}_{\mathbf{P}^1_{\bar{k}}} (\mathcal{O}_{\mathbf{P}^1_{\bar{k}}} \oplus \mathcal{O}_{\mathbf{P}^1_{\bar{k}}}(N)), \quad N \geqslant 0,$$

is the standard scroll.

Now suppose that $f: F \longrightarrow C$ does not have a section over k. Then by the Noether-Tsen lemma $\bar{f}: \bar{F} \longrightarrow \bar{C}$ has a section, and by the above argument, there is an isomorphism $\bar{F} \simeq \mathbf{F}_N$ for some $N \geqslant 0$.

Let $N \geqslant 1$, and let $S \subset \mathbf{F}_N$ be the image of the canonical section: $(S \cdot S) = -N$ and $(S \cdot K_{\mathbf{F}_N}) = 2N - 2$. Since the curve S with negative intersection is unique on $\mathbf{F}_N$, there must exist a curve X on F which is geometrically irreducible and reduced over k, such that $X \otimes \bar{k} = qS$. From the irreducibility and geometrical connectedness of X, $H^0(X, \mathcal{O}_X)$ must be a purely radicial field extension of k, with $q \geqslant \dim H^0(X, \mathcal{O}_X)$. From the genus formula (1.3) we get at once that $q = 1$; that is, in this case $F \longrightarrow C$ must have a section. Hence there remains the case $N = 0$, $F \simeq C \times C'$, where C' does not have k-points.

It is known that $(\Omega_{\mathbf{F}_N} \cdot \Omega_{\mathbf{F}_N}) = 8$, and hence $(\Omega_F \cdot \Omega_F) = 8$.

(3) If f is not smooth, it cannot have a section. For if $S = s(C)$, then $(\bar{S} \cdot \bar{F}_{\bar{t}}) = 1$. Suppose that $\bar{F}_{\bar{t}} = \bar{X}_1 + \bar{X}_2$ is a degenerate fiber; then, since $(\bar{S} \cdot \bar{X}_1) = (\bar{S} \cdot \bar{X}_2)$, $(\bar{S} \cdot \bar{F}_{\bar{t}}) > 1$, a contradiction. Hence the generic fiber F_η does not have points of degree 1 over $k(\eta)$,

and Pic F_η is generated by $\Omega_F^{-1} \otimes \mathcal{O}_{F_\eta}$, a sheaf of degree 2. In the exact sequence (1.7) Pic $F/C \simeq$ Pic $F_\eta \simeq \mathbf{Z}$, and hence

$$\operatorname{Pic} F = f^* \operatorname{Pic} C \oplus \mathbf{Z} \cdot \Omega_F^{-1}.$$

Each component of a geometric degenerate fiber is an exceptional curve of the first kind on $\bar{F}$ by Lemma 6. Contracting one of these components for each such fiber (there are r of them), we get one of the scrolls $\mathbf{F}_N$ (see (2) of Theorem 3). It follows at once that rk Pic $\bar{F} = 2 + r$ (since rk Pic $\mathbf{F}_N = 2$), and $(\Omega_{\bar{F}} \cdot \Omega_{\bar{F}}) = 8 - r$; if $(\Omega_F \cdot \Omega_F)$ is odd then $r = \Sigma \deg t_i$ is odd; that is, C would have a divisor of degree h.c.f.$(r, 2p(C) - 2) = 1$. By the Riemann-Roch theorem C would then have a k-point, and $C \simeq \mathbf{P}_k^1$.

(4) First of all we remark that $F \longrightarrow C$ and $F' \longrightarrow C$ are birationally equivalent over C if and only if their generic fibers F_η and F'_η are isomorphic over $k(\eta)$. Hence by the classical theory the corresponding quaternion algebras A_η and A'_η over $k(\eta)$ are isomorphic.

We consider first the case that $f: F \longrightarrow C$ is smooth. Then according to Grothendieck's definition [5], $F \longrightarrow C$ is a Severi-Brauer scheme over C of relative dimension 1, and to it there corresponds an Azumaya algebra of rank 4 over C, having stalk A_η at the generic point. The correspondence between Azumaya algebras and Severi-Brauer schemes is bijective [5], so that we only need verify that the Azumaya algebras which extend A_η are exactly the maximal orders of A_η over C. The general case of this is proved in [4].

A generalization of the previous correspondence to nonsmooth morphisms of relative dimension 1 with a 2-dimensional base C is given in [2]. When C is 1-dimensional the arguments given in [2] can be simplified a little, and we reproduce them very briefly here. Let A be any maximal order in A_η over C. Then (as in the smooth case) one can associate to A the C-scheme of its left ideals $X \longrightarrow C$ (the closed subscheme of the Grassmannian over C representing the functor $X(C') = \{$left ideals I in $A \otimes_{\mathcal{O}_C} \mathcal{O}_{C'}$ of rank 2$\}$). A_η has only a finite number of ramification points $t_1, \dots, t_s \in C$, and outside these A_η extends to an Azumaya algebra. Clearly the t_i are contained in the set of degeneracy of the morphism $f: F \longrightarrow C$. We want to elucidate the structure of the scheme $X \longrightarrow C$ locally in a neighborhood of any point $t \in C$. If t is not a ramification point, then, because A is a maximal order, $A \otimes \mathcal{O}_t$ is locally an Azumaya algebra, and hence the corresponding fiber X_t is smooth and of genus 0 (as the fiber of the Severi-Brauer scheme $X \otimes \mathcal{O}_t \longrightarrow \operatorname{Spec} \mathcal{O}_t$).

Now suppose that $t \in \{t_1, \dots, t_s\}$. Then, as is well known, a maximal order in A_η over the discrete valuation ring $\mathcal{O}_t$ can be given by means of a basis $(1, i, j, ij)$ with the multiplication table

$$\begin{aligned} &i^2 = \alpha, \quad j^2 = \beta t, \quad ij = -ji, \quad \text{if} \quad \operatorname{char} k \neq 2, \\ &i^2 = i + \alpha, \quad j^2 = \beta t, \quad ij = ji + j, \quad \text{if} \quad \operatorname{char} k = 2, \end{aligned} \tag{3.1}$$

where t is a local parameter at the point t, α and $\beta \in \mathcal{O}_t$ are invertible, and

$$\begin{aligned} &x^2 - \alpha \not\equiv 0 \pmod{t}, \quad \text{if} \quad \operatorname{char} k \neq 2, \\ &x^2 + x + \alpha \not\equiv 0 \pmod{t}, \quad \text{if} \quad \operatorname{char} k = 2, \end{aligned} \tag{3.2}$$

for any $x \in \mathcal{O}_t$. Conversely for any $\alpha, \beta \in \mathcal{O}_t$ satisfying (3.2), the relations (3.1) define a maximal $\mathcal{O}_t$-order in some nontrivial quaternion algebra. The elements α and β are invertible in some neighborhood of t, so that our maximal order A is given by the relations (3.1) in some neighborhood $U \ni t$. From (3.1) one easily establishes the equations defining the scheme of left ideals $X_U \longrightarrow U$ as a closed subscheme of $\mathbf{P}^2_U$:

$$\begin{aligned} x_0^2 - \alpha x_1^2 - \beta t x_2^2 = 0, \quad &\text{if} \quad \operatorname{char} k \neq 2, \\ x_0^2 + x_0 x_1 + \alpha x_1^2 + \beta t x_2^2 = 0, \quad &\text{if} \quad \operatorname{char} k = 2, \end{aligned} \tag{3.3}$$

where (x_0, x_1, x_2) are homogeneous coordinates in $\mathbf{P}^2_U$. From (3.3) one sees at once that X_U is a smooth surface over k, since $k(t)/k$ is separable (see the corollary to Lemma 6) and all the fibers of $X_U \longrightarrow U$ are smooth conics, except for X_t, which splits over the quadratic extension $k(t)(\sqrt{\alpha \bmod t})$ into a pair of distinct lines.

Thus for any maximal $\mathcal{O}_C$-order A, the scheme of its left ideals X is a complete smooth surface over k, and the morphism $X \longrightarrow C$ satisfies all the properties in II of Theorem 1.

For the proof that the correspondence $A \mapsto C$ is bijective, note that all maximal orders in A_η are C-forms in the etale topology of a certain standard order A_0 in the trivial quaternion algebra, namely, that having the local representation (3.1) with $\alpha = \beta = 1$. Similarly, if $X_0 \longrightarrow C$ is the scheme of left ideals of the order A_0, then $X \longrightarrow C$ is locally isomorphic in the etale topology to $X_0 \longrightarrow C$. Thus, according to the standard cohomological description of forms, to have a bijection it is enough to show that the natural morphism of the etale sheaves of automorphisms $\operatorname{Aut} A_0 \longrightarrow \operatorname{Aut} X_0$ is an isomorphism. Outside the critical points this is well known [5]; at a critical point this can be verified directly using the explicit forms (3.1) and (3.3). We omit this simple verification (see [4]).

It is clear that, in the bijective correspondence $A \mapsto X$, corresponding to the maximal orders of A_η we have precisely the standard forms $F' \longrightarrow C$ whose generic fibers are isomorphic to F_η. This completes the proof of Theorem 3.

REMARK. Part (4) of Theorem 3 gives a description of the relative minimal models of function fields $k(F)/k(C)$ of genus 0 over a smooth algebraic curve C (compare [18]).

PROOF OF THEOREM 4. Let $f\colon F \longrightarrow C$ be a standard form of family II and suppose that $(\Omega_F \cdot \Omega_F) = 8$. Then, by (2) and (3) of Theorem 3, f is smooth, and $\bar{F} \simeq \mathbf{F}_N$ for some $N \geqslant 0$. It is well known (see [1]) that for $N \neq 1$ all the $\mathbf{F}_N$ are minimal, and hence F is minimal. As in the proof of (2) of Theorem 3, if $\bar{F} \simeq \mathbf{F}_1$ then F contains a curve (the section X), which can be contracted by the Castelnuovo-Enriques criterion. Under this contraction F will be a surface F' with $\operatorname{Pic} F' \simeq \mathbf{Z}$, and it can easily be deduced from the proof of Theorem 1 that $F' \simeq \mathbf{P}^2_k$.

Now suppose that $(\Omega_F \cdot \Omega_F) \neq 8$. Then the morphism $f\colon F \longrightarrow C$ is not smooth, and by Theorem 3

$$\operatorname{Pic} F = f^* \operatorname{Pic} C \oplus \mathbf{Z} \cdot \Omega_F^{-1}.$$

Suppose that F is not minimal. Then there must be some exceptional curve of the first kind $X \subset F$; let $a, b \in \mathbf{Z}$ be such that $X \sim aF_t - bK_F$, where F_t is some generator of $f^* \operatorname{Pic} C$ (the class of a fiber over some $t \in C$), and K_F is the canonical divisor. Since there are no contractible curves in the fibers of f, it follows that

$$0 < (X \cdot F_t) = 2 \deg t b;$$

that is, $b > 0$. We have

$$(X \cdot X) = 4ab \deg t + b^2 (\Omega_F \cdot \Omega_F) = -m,$$
$$(X \cdot K_F) = -2a \deg t - b (\Omega_F \cdot \Omega_F) = -m, \tag{3.4}$$

where $m \geqslant 1$ is the number of geometric components of $\bar{X}$. Set $n = (\Omega_F \cdot \Omega_F)$. Then from (3.4) we get

$$-nb^2 + 2mb + m = 0. \tag{3.5}$$

For $n \leqslant 0$ this equation does not have solutions in natural numbers m and b, and hence the corresponding surfaces are minimal. If $1 \leqslant n \leqslant 9$ we solve (3.5) to get

$$b_{1,2} = \frac{m \pm \sqrt{m(m+n)}}{n}.$$

In order for integer solutions to exist, $m(m + n)$ must be the square of an integer; that is, the following equation must be solvable in natural numbers:

$$m(m+n) = r^2, \quad 1 \leqslant n \leqslant 9. \tag{3.6}$$

Let d = h.c.f.$(m, m + n)$; then we must have $m \leqslant (n - d)/2$. From this it is easy to obtain all solutions of (3.6), and so all solutions of (3.5). We have

n	m	a	b	$\deg t$
3	1	−1	1	1
5	4	−3	2	1
6	2	−2	1	1
6	2	−1	1	2

It remains to verify that exceptional curves with the indicated numerical characteristics actually exist on the corresponding surfaces. Since

$$K_F - X \sim -aF_t + (b+1) K_F$$

and

$$(K_F - X \cdot F_t) = -2(b+1) \deg t < 0,$$

we have

$$\dim H^0(F, \Omega_F \otimes \mathcal{O}_F(-X)) = \dim H^2(F, \mathcal{O}_F(X)) = 0.$$

By the Riemann-Roch theorem

$$\dim H^0(F, \mathcal{O}_F(X)) \geqslant \frac{-m+m}{2} + 1 = 1.$$

Hence all the numerical types indicated in the table actually occur, and surfaces F with $n = 3, 5$ and 6 are not minimal. Since rk Pic $F = 2$, F can contain more than 1 contractible curve X. After the contraction of X we obtain a surface F' of family I for which we have $(\Omega_{F'} \cdot \Omega_{F'}) = 4, 9$ or 8 respectively. The theorem is proved.

PROOF OF THEOREM 5. (1) Let $n = (\Omega_F \cdot \Omega_F)$. Let us check that on a surface F with $n = 3$, 5 or 6, Ω_F^{-1} is ample. By the numerical criterion of ampleness we have to show that F does not contain irreducible reduced curves Y such that $(Y \cdot -K_F) \leqslant 0$. Since $\dim H^0(F, \Omega_F^{-1}) \geqslant n + 1$, if such a curve were to exist it would have to be either a fixed component of $|\Omega_F^{-1}|$, or a curve with $(Y \cdot K_F) = 0$ and $(Y \cdot Y) < 0$.

If Y is a fixed component, then $Y \sim -aF_t - K_F$, where $a \geqslant 1$. We have

$$(Y \cdot -K_F) = -2a \deg t + n \leqslant 0,$$
$$(Y \cdot Y) = -4a \deg t + n,$$
$$(Y \cdot F_t) = 2 \deg t,$$
$$p(Y) = \frac{(Y \cdot Y + K_F)}{2} + 1 = -a \deg t + 1 \leqslant -\frac{n}{2} + 1.$$

From the third equation it follows that $\overline{Y}$ can have at most 2 geometric components, or 1 component of multiplicity 2. If $n \geqslant 3$ then the final inequality implies that Y is geometrically reducible or multiple, which contradicts the second equation for odd n. On the other hand, if the number of geometric components or the multiplicity is not greater than 2 then from the formula for the genus we get $p(Y) \geqslant -1$, and for $n \geqslant 5$ this contradicts the fourth inequality.

Similarly, if $Y \sim -\alpha F_t - \beta K_F$, with $\beta \geqslant 1$, and $(Y \cdot K_F) = 0$, then

$$(Y \cdot K_F) = 2\alpha \deg t - \beta n = 0, \qquad (Y \cdot Y) = -2\alpha\beta \deg t < 0.$$

Let $Y = q \Sigma_1^m \overline{Y}_i$, where the $\overline{Y}_i$ are connected components and q is the multiplicity. Since $(\overline{Y}_i \cdot \overline{K}_F) = 0$ and $(\overline{Y}_i \cdot \overline{Y}_i) < 0$, we have $(\overline{Y}_i \cdot \overline{Y}_i) = -2$ and $p(\overline{Y}_i) = 0$ (by (1.2)). Hence $(Y \cdot Y) = -2q^2 m$, and $(F_t \cdot Y) = 2\beta \deg t = \deg t \cdot m \cdot q \cdot s$, where $s = (\overline{Y}_i \cdot F_{\overline{t}})$ is the number of points of intersection of the geometric fiber with any of the connected components $\overline{Y}_i$. Making the corresponding substitutions, we get $2q^2 m = m^2 q^2 s^2 n/4$, or $ms^2 n = 8$, which is only possible for $n = 1, 2, 4$ or 8. Thus the first assertion is proved.

(2) If $n = 8$ then there exists an N such that $\overline{F} \simeq \mathbf{F}_N$. Let $S \subset \mathbf{F}_N$ be the section with $(S \cdot S) = -N$; then $(S \cdot -K_{\mathbf{F}_N}) = -N + 2$, so that for $N \geqslant 2$ $\Omega_{\mathbf{F}_N}^{-1}$, and hence also Ω_F^{-1}, cannot be ample. It is trivial to check that Ω_F^{-1} is ample, even very ample, if $N = 0$ or 1.

(3) Let $n = 1$, 2 or 4, and suppose that F has a further representation $h: F \longrightarrow C'$ in the standard form II. Then for the numerical characters of the geometric fiber $F_{\overline{t}'} = h^{-1}(\overline{t}')$ for $\overline{t}' \in \overline{C}'$ we get immediately the single possibility

$$F_{\overline{t}'} \sim \begin{cases} -F_{\overline{t}} - 4K_F, & n = 1, \\ -F_{\overline{t}} - 2K_F, & n = 2, \\ -F_{\overline{t}} - K_F, & n = 4, \end{cases}$$

where $F_{\overline{t}}$ is the geometric fiber of f, $\overline{t} \in \overline{C}$. Now using the same arguments as in (1), and also the Riemann-Roch theorem, we see that these possibilities are realized if and only if Ω_F^{-1} is ample. Note that from the property (b) of del Pezzo surfaces (see p. 30) one sees that for $n = 1$ and 2 F has an involution which, as one easily checks, interchanges the two standard morphisms f and h.

If $n = 7$, then $\bar{F}$ is not minimal, and we have either a contraction $\bar{F} \longrightarrow \mathbf{F}_N$ of one exceptional curve, or $\bar{F} \longrightarrow \mathbf{P}^2_{\bar{k}}$ of two. In either case $\bar{F}$ contains only 3 curves with negative selfintersection, and an easy analysis shows that F must contain an exceptional curve of the first kind defined over some inseparable extension of k. Then, as in the proof of (2) of Theorem 3, we see that it is in fact defined over k, and can thus be contracted. This concludes the proof.

Theorems 5, 4 and 1 immediately give us the following minimality criterion for del Pezzo surfaces, which generalizes a criterion of Segre for cubic surfaces.

COROLLARY. *Let F be a del Pezzo surface, and let* $(\Omega_F \cdot \Omega_F) = 3, 5, 6$ *or* 9. *Then F is minimal if and only if* Pic $F \simeq \mathbf{Z}$. *If* $(\Omega_F \cdot \Omega_F) = 1, 2$ *or* 4 *then F is minimal if and only if either* Pic $F \simeq \mathbf{Z}$, *or* Pic $F \simeq \mathbf{Z} \oplus \mathbf{Z}$ *and F belongs to family* II. *In the remaining cases* $(\Omega_F \cdot \Omega_F) = 7$ *or* 8, *F is minimal only if* $(\Omega_F \cdot \Omega_F) = 8$ *and either* Pic $F \simeq \mathbf{Z}$, *or* $F \simeq C \times C'$, *where C and C' are smooth curves of genus* 0.

§4. Minimal rational G-surfaces

DEFINITION (see [11]). A complete smooth surface F over a field k, together with a finite group G acting on F by k-automorphisms, is called a *G-surface*. A morphism $f: F \longrightarrow F'$ of G-surfaces is called a *G-morphism* if $g \circ f = f \circ g$ for all $g \in G$. If any birational G-morphism $F \longrightarrow F'$ of smooth complete G-surfaces is a G-isomorphism, F is called a *G-minimal surface.*

A G-invariant curve $X \subset F$ is said to be *G-irreducible* if $X = \Sigma_1^r X_i$, where X_i are distinct irreducible curves on F belonging to a single G-orbit.

Consider the exact sequence of G-modules

$$1 \to k(F)^* \to \text{Div } F \to \text{Pic } F \to 1$$

and the corresponding sequence of group cohomology

$$1 \to k(F)^{*G} \to (\text{Div } F)^G \to (\text{Pic } F)^G \to \dots .$$

Let $P(F)$ denote the image of $(\text{Div } F)^G$ in $(\text{Pic } F)^G$; this is the subgroup of Pic F generated by the classes of invertible sheaves of the form $\mathcal{O}_X(F)$, where X is a G-invariant divisor.

Let $L \in P(F)$ be an invertible sheaf; then there is a natural structure of G-modules on the vector spaces $H^i(F, L)$.

One sees easily that $\Omega_F \in P(F)$. In fact, $\Omega_F \simeq \mathcal{O}_F(K)$, where we can choose K to be the divisor of some G-invariant differential 2-form.

Note, finally, that the pairing Pic $F \times$ Pic $F \longrightarrow \mathbf{Z}$ defined by the intersection number is G-invariant if F is a G-surface.

PROOF OF THEOREM 1G. We will analyse step-by-step the proof of Theorem 1, and pause only over those points which require further considerations.

Step 1. Lemmas 1 and 2 do not depend on the G-structure.

Step 2. In Lemma 3 Pic F must be replaced by $P(F)$, and $L \in P(F)$; property (3) must be replaced by

(3G) L cannot be represented in the form $L_1 \otimes L_2$, with $L_i \in P(F)$, $L_i \neq \mathcal{O}_F$ and $\dim H^0(F, L_i) \geqslant 1$ for $i = 1, 2$.

In the proof H can be taken as the G-orbit of any very ample sheaf.

Step 3. In the proof of Lemma 4(a) the linear system $|L|$ cannot have a fixed component, since otherwise the fixed component would obviously be a G-curve, so that L would not satisfy $(3G)$. For the same reason X must be G-irreducible and G-reduced. If X is also k-irreducible, the subsequent argument goes through without any change. If $X = \Sigma X_i$, then X_i is reduced and irreducible for each i, and $(X_i \cdot X_i) < 0$, since $(X \cdot X) < 0$. Furthermore the sheaf $L_i = \mathcal{O}_F(X_i)$ satisfies all the conditions of Lemma 4 without G-structure, and so a contradiction is reached in the same manner.

Assertion 4(b) goes through without change; one supposes throughout, unless the contrary is stated, that all the invertible sheaves occurring belong to $P(F)$.

Step 4. In Lemma 5 and subsequently, isomorphism must be taken to mean G-isomorphism. The morphism $f\colon F \longrightarrow \operatorname{Proj} R = C$ constructed in the proof of Theorem 1 is obviously a G-morphism; the G-structure on C is induced by the action of G on the ring R. In Lemma 4 the first assertion does not depend on the G-structure. In the second assertion as to the irreducibility of the fibers, one has to understand the G-irreducibility of fibers over a G-point $t \in G$ (where a G-point is taken to be a G-orbit of any point). The remaining assertions of the lemma are independent of the G-structure.

Finally, the assertion about $P(F)$ is obtained by means of the same arguments, using the fact that all the terms of the spectral sequence are G-modules. From (1.7) we get the sequence

$$0 \to \operatorname{Pic}^G C \to \operatorname{Pic}^G F \to \operatorname{Pic}^G(F/C) \to \dots,$$

whence $\operatorname{Pic}^G F \simeq \mathbf{Z} \oplus \mathbf{Z}$, and one sees easily that $P(F) \subset \operatorname{Pic}^G F$ is a subgroup of finite index. This completes the proof of Theorem $1G$.

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