On cubic surfaces with a rational line

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Abstract. We report on our project to construct non-singular cubic surfaces over $\mathbb{Q}$ with a rational line. Our method is to start with degree 4 Del Pezzo surfaces in diagonal form. For these, we develop an explicit version of Galois descent.

Mathematics Subject Classification (2010). Primary 14J26; Secondary 14G25, 11G35.

Keywords. Cubic surface, Degree 4 Del Pezzo surface, Explicit Galois descent.

1. Introduction

1.1. The configuration of the 27 lines upon a smooth cubic surface is highly symmetric. The group of all permutations respecting the canonical class as well as the intersection pairing is isomorphic to the Weyl group $W(E_6)$ of order 51 840.

When S is a cubic surface over $\mathbb{Q}$, the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ operates on the 27 lines. This yields a subgroup $G \subseteq W(E_6)$. It is an open problem whether each of the 350 conjugacy classes of subgroups of $W(E_6)$ may be realized by a cubic surface over $\mathbb{Q}$.

Exactly 172 of the 350 conjugacy classes fix a line. We constructed examples of cubic surfaces over $\mathbb{Q}$ realizing each of these subgroups. The goal of this note is to report on our investigations.

Remark 1.2. The analogous question for Del Pezzo surfaces of degree 4 is somewhat easier as it leads to subgroups of $W(D_5)$. B. È. Kunyavskij, A. N. Skorobogatov, and M. A. Tsfasman [4] showed that every subgroup of $W(D_5)$ may be realized by a surface defined over $\mathbb{Q}$.

The first author was supported in part by the Deutsche Forschungsgemeinschaft (DFG) through a funded research project.

2. Constructions

2.1. Cubic surfaces with a rational line are closely related to Del Pezzo surfaces of degree 4. Indeed, blowing down the line leads to a degree 4 Del Pezzo surface having a rational point. On the other hand, blowing up a rational point on a degree 4 Del Pezzo surface yields a cubic surface with a rational line. These two constructions may easily be made explicit.

Constructions 2.2 (Cubic surfaces versus Del Pezzo surfaces of degree 4). Let a base field K be fixed once and for all.

i) For two linear forms l_0, l_1, suppose that the line $l_0 = l_1 = 0$ is contained in the cubic surface S given by $F(x_0, \ldots, x_3) = 0$. Then, F may be written as $F = l_0 q_0 + l_1 q_1$ for quadratic forms q_0 and q_1. The corresponding degree 4 Del Pezzo surface V is given by $q_0 + l_1 x_4 = q_1 - l_0 x_4 = 0$.

ii) On the other hand, let a Del Pezzo surface V of degree 4 be given by $Q_0(x_0, \ldots, x_4) = Q_1(x_0, \ldots, x_4) = 0$. If $(0:0:0:0:1) \in V$ then Q_0 and Q_1 may be written as $Q_0 = q_0 + l_0 x_4$ and $Q_1 = q_1 + l_1 x_4$ for q_0, q_1 quadratic forms and l_0, l_1 linear forms in $x_0, \ldots, x_3$, only. The corresponding cubic surface S is given by $q_0 l_1 - q_1 l_0 = 0$.

Remarks 2.3. a) These two constructions are inverse to each other.

b) One may start construction ii) as well with arbitrary generators of the pencil spanned by Q_0 and Q_1.

Fact 2.4. *Let A be a symmetric matrix representing the quadratic form $q_0|_{l_0=0}$. If the eigenvalues of A are z_1, z_2, z_3 then there is a symmetric matrix representing Q_0 with eigenvalues $(-1), 1, z_1, z_2, z_3$.*

Corollary 2.5. i) *In particular, Q_0 is of rank <5 if and only if $q_0|_{l_0=0}$ is of rank <3. Hence, the five degenerate quadratic forms in the pencil $[Q_0, Q_1]$ are in one-to-one correspondence with the five tritangent planes through the line considered.*

ii) *If the eigenvalues of a symmetric matrix representing Q_0 are $0, Z_1, \ldots, Z_4$ then "$l_0 = 0$" is a tritangent plane on S. The conic, defined by S on this plane, splits into two lines over the field $K(\sqrt{Z_1 Z_2 Z_3 Z_4})$.*

Example. Consider the case that $Q_0 := a_0 x_0^2 + \ldots + a_4 x_4^2$ and $Q_1 := b_0 x_0^2 + \ldots + b_4 x_4^2$ are diagonal forms over the field K. Then, the five tritangent planes correspond to the points $((-b_i) : a_i) \in \mathbf{P}^1$ as $(-b_i Q_0 + a_i Q_1)$ is degenerate. The conics split over the fields

$$K\big(\sqrt{\textstyle\prod_{j \neq i} (-b_i a_j + a_i b_j)}\big)$$

for $i = 0, \ldots, 4$. Observe that the product of the five radicands is a perfect square.

On the corresponding cubic surface, all 27 lines are defined over

$$L = K\big(\sqrt{\textstyle\prod_{j \neq 0} (-b_0 a_j + a_0 b_j)}, \ldots, \sqrt{\textstyle\prod_{j \neq 4} (-b_4 a_j + a_4 b_j)}\big)\,.$$

Indeed, the subgroup of $W(E_6)$ stabilizing a line is clearly of order $51840/27 = 1920$. It is actually the semi-direct product $T \rtimes S_5$, where

$T \subset (\mathbb{Z}/2\mathbb{Z})^5$ is the subgroup of order 16 formed by the elements having an even number of components equal to 1. As $\mathrm{Gal}(\overline{\mathbb{Q}}/L)$ stabilizes not only the five tritangent planes but also the lines on them, it must act through the trivial subgroup of $T \rtimes S_5$.

Construction 2.6 (Explicit Galois descent). Let A be a commutative étale algebra of degree 5 over $\mathbb{Q}$ and $\iota_0, \ldots, \iota_4 \colon A \to \mathbb{C}$ be the five embeddings.

i) For general $a,b \in A$, the equations

$$\iota_0(a)x_0^2 + \cdots + \iota_4(a)x_4^2 = \iota_0(b)x_0^2 + \cdots + \iota_4(b)x_4^2 = 0$$

define a Del Pezzo surface V of degree 4 over $\overline{\mathbb{Q}}$.

ii) Let l be a linear form in five variables with coefficients in A. Then, by symmetry, the quadratic forms $\iota_0(a)(l^{\iota_0})^2 + \cdots + \iota_4(a)(l^{\iota_4})^2$ and $\iota_0(b)(l^{\iota_0})^2 + \cdots + \iota_4(b)(l^{\iota_4})^2$ have rational coefficients. If $l^{\iota_0}, \ldots, l^{\iota_4}$ are linearly independent then we have a Del Pezzo surface V_0 of degree 4 over $\mathbb{Q}$ such that its base change to $\overline{\mathbb{Q}}$ is isomorphic to V.

Remarks 2.7. a) This construction is analogous to [2, Theorem 6.1].

b) The five tritangent planes on V_0 correspond to the points $((-\iota_i(b)) : \iota_i(a)) \in \mathbf{P}^1$. Hence, the Galois operation on them is the same as that on the embeddings ι_i.

c) When $a \neq 0$, the conic on the tritangent plane corresponding to $((-\iota_i(b)) : \iota_i(a))$ splits into two lines over the field

$$\mathbb{Q}\Big(\iota_i(-b/a), \sqrt{\prod_{j \neq i} (-\iota_i(b)\iota_j(a) + \iota_i(a)\iota_j(b))}\Big)\,. \tag{2.1}$$

The radicand may be rewritten as $\mathrm{N}(a)\, \iota_i(a^3\, \delta_{A/\mathbb{Q}}(-b/a))$, where $\delta_{A/\mathbb{Q}}$ denotes the different of an element of A.

2.8. Thus, given a subgroup $G \subseteq T \rtimes S_5$, there is the following strategy to construct a cubic surface S over $\mathbb{Q}$ such that $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ operates via G on the 27 lines.

Strategy. i) Find a number field K, normal over $\mathbb{Q}$, such that $\mathrm{Gal}(K/\mathbb{Q}) \cong G$. Identify the normal subextension $K' \subseteq K$ such that $\mathrm{Gal}(K'/\mathbb{Q})$ is the image G' of G in S_5 [5].

ii) Find five elements $r_0, \ldots, r_4 \in K'$ with the properties below.

$r_0, \ldots, r_4$ are permuted by G' exactly via the embedding $G' \subseteq S_5$. Further, the square roots $\pm\sqrt{r_0}, \ldots, \pm\sqrt{r_4}$ are elements of K and acted upon by G according to the embedding $G \subseteq T \rtimes S_5$.

Put $p(T) := (T - r_0) \cdot \ldots \cdot (T - r_4)$ and $A := \mathbb{Q}[T]/(p)$. This is a commutative étale algebra of degree 5 over $\mathbb{Q}$ with a distinguished element $r := (T \bmod (p))$.

iii) Choose $x \in A$ and put $d := \delta_{A/\mathbb{Q}}(x)$. Set $a := dr$ and $b := -xa$.

iv) Execute Construction 2.6 for $a, b \in A$. On the Del Pezzo surface V_0 found, search for a $\mathbb{Q}$-rational point. If none is found then go back to step iii). Otherwise, determine the cubic surface S.

Remarks 2.9. a) The properties required in ii) imply $\sqrt{r_0} \cdot \ldots \cdot \sqrt{r_4} \in \mathbb{Q}$. I.e., $\mathrm{N}(r)$ is a perfect square.

b) The construction yields $\mathrm{N}(a)a^3\,\delta_{A/\mathbb{Q}}(-b/a) = \mathrm{N}(a)(d^2r)^2r$. As the product of the five radicands in (2.1) is a square, the norm of a is a perfect square automatically.

3. Examples

3.1. There are 172 conjugacy classes of subgroups of $W(E_6)$ that fix a line. We constructed examples for each such group.

Actually, 81 of the 172 classes also stabilize a double-six and 49 of the 172 classes stabilize a pair of Steiner trihedra. 34 classes do both. Thus, examples for 96 of the 172 conjugacy classes had been constructed before [2, 3]. The remaining 76 classes were of interest.

After naive trials and an extensive search through surfaces with small coefficients, only six of the 76 classes remained open. For these, we applied Strategy 2.8.

Remark 3.2. In Strategy 2.8, we regularly run into reiteration, because there were no $\mathbb{Q}$-rational points on the Del Pezzo surfaces of degree 4.

3.3. The list containing our examples of cubic surfaces is available on the second author's web page at `http://www.uni-math.gwdg.de/jahnel/Arbeiten/Kub_Fl/list_rat_ger.txt`. The numbering of the subgroups is that created by `GAP`, version 4.4.12.

Example. As a conclusion, let us show how Strategy 2.8 works on a particular example. We consider the subgroup of number 107.

Abstractly, this is a group G of order 16. Its center is isomorphic to the Klein four-group. The operation on the 27 lines causes orbits of lengths 1, 2, 4, 4, and 16. On the two orbits of size four, G acts via two different quotients, both isomorphic to the dihedral group D_4 of order eight. The operation on the five tritangent planes through the rational line is via a quotient G' of order four. The orbits are of sizes 1, 2, and 2.

i) An example of a field with Galois group G is the composite $K := K_1K_2$ of $K_1 := \mathbb{Q}\big(\sqrt{3 \pm \sqrt{3}}\big)$ and $K_2 := \mathbb{Q}\big(\sqrt{-9 \pm \sqrt{6}}\big)$. Then, the subfield corresponding to G' is $K' = \mathbb{Q}(\sqrt{3}, \sqrt{6}) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Observe that both fields K_1 and K_2 contain K'. Further, both are extensions of $\mathbb{Q}$ of type D_4.

ii) Thus, we chose $r_0, \ldots, r_4$ to be $2, 3 \pm \sqrt{3}$, and $-9 \pm \sqrt{6}$. This yields

$$p(T) = (T-2)[(T-3)^2 - 3][(T+9)^2 - 6]\,.$$

iii) We worked with $x := r = (T \bmod (p))$.

iv) The biggest coefficient of the resulting del Pezzo surface V_0 is 524 391 211 895 464. An isomorphic surface is given by the equations

$$\begin{aligned} & 4x_0^2 + 10x_0x_1 + 20x_0x_2 - 112x_0x_3 - 134x_0x_4 + 7x_1^2 - 26x_1x_2 - 134x_1x_3 \\ & - 148x_1x_4 - 2x_2^2 + 140x_2x_3 - 2x_2x_4 + 10x_3^2 - 38x_3x_4 - 323x_4^2 \\ = {} & 47x_0^2 - 18x_0x_1 + 10x_0x_2 - 188x_0x_3 - 178x_0x_4 + 63x_1^2 - 22x_1x_2 + 376x_1x_3 \\ & - 86x_1x_4 + 71x_2^2 - 580x_2x_3 + 146x_2x_4 - 364x_3^2 - 296x_3x_4 - 21x_4^2 = 0\,. \end{aligned}$$

Here, a point search in `magma` with an initial height limit of 100 shows 14 rational points. Blowing up $(8 : -13 : 4 : 2 : -3)$ leads to a cubic surface with coefficients up to 3 838 320. Reembedding gives us the final result, the cubic surface V with the equation

$$\begin{aligned} & 2x^2y + 6x^2z - 4xy^2 + 6xyz + 4xyw - 10xz^2 - 4xzw - 7xw^2 + 2y^3 \\ & - 9y^2z - 4y^2w + 4yz^2 - 26yzw + 6yw^2 + z^3 + 10z^2w - 7zw^2 - 5w^3 = 0\,. \end{aligned}$$

Remark 3.4. The rational line on V connects $(5 : 0 : 0 : -7)$ with $(0 : 5 : 10 : 2)$.

Remark 3.5. There are actually a few more particularities characterizing the subgroup of number 107.

a) First of all, the two D_4 extensions K_1 und K_2 become cyclic over the same quadratic field $\mathbb{Q}(\sqrt{2})$.

b) On the other hand, over $\mathbb{Q}(\sqrt{3})$ and $\mathbb{Q}(\sqrt{6})$, they are of Kleinian type. However, there is yet another oddity. While $\mathrm{Gal}(K_1/\mathbb{Q}(\sqrt{3}))$ operates on the corresponding four lines via two disjoint two-cycles, $\mathrm{Gal}(K_2/\mathbb{Q}(\sqrt{3}))$ acts on its orbit by double-transpositions. Over $\mathbb{Q}(\sqrt{3})$ instead of $\mathbb{Q}(\sqrt{6})$, the situation is vice versa.

To realize such a behaviour, it was essential to choose r_1 in $\mathbb{Q}(\sqrt{3})$ fulfilling $\mathrm{N}(r_1) \in 6(\mathbb{Q}^*)^2$ and r_3 in $\mathbb{Q}(\sqrt{6})$ such that $\mathrm{N}(r_3) \in 3(\mathbb{Q}^*)^2$.

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