LUIGI CREMONA AND CUBIC SURFACES

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Abstract. We discuss the contribution of Luigi Cremona to the early development of the theory of cubic surfaces.

1. A brief history

In 1911 Archibald Henderson wrote in his book [Hen]

"While it is doubteless true that the classification of cubic surfaces is complete, the number of papers dealing with these surfaces which continue to appear from year to year furnish abundant proof of the fact that they still possess much the same fascination as they did in the days of their discovery of the twenty-seven lines upon the cubic surface."

It is amazing that a similar statement can be repeated almost a hundred years later. Searching in MathSciNet for "cubic surfaces" and their close cousins "Del Pezzo surfaces" reveals 69 and 80 papers published in recent 10 years.

Here are some of the highlights in the history of classical research on cubic surfaces before the work of Cremona. A good source is Pascal's Repertorium [Pas].

1849: Arthur Cayley communicates to George Salmon that a general cubic surface contains a finite number of lines. Salmon proves that the number of lines must be equal to 27. Salmon's proof is presented in Cayley's paper [Cay]. In the same paper Cayley shows that a general cubic surface admits 45 tritangent planes, i.e. planes planes which intersect the surface along the union of three lines. He gives a certain 4-parameter family of cubic surfaces for which the equations of tritangent planes can be explicitly found and their coefficients are rational functions in parameters.

In a paper published in the same year and in the same journal [Sal1] Salmon proves that not only a general but any nonsingular surface contains exactly 27 lines. He also finds the number of lines on 11 different types of singular surfaces.

The discovery of 27 lines on a general cubic surface can be considered as the first non-trivial result on algebraic surfaces of order higher than 2. In fact, it can be considered as the beginning of modern algebraic geometry.

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1851: John Sylvester claims without proof that a general cubic surface can be written uniquely as a sum of 5 cubes of linear forms [Syl]:

$$F_3 = L_1^3 + L_2^3 + L_3^3 + L_4^3 + L_5^3.$$

This was proven ten years later by Alfred Clebsch [Cle3]. The union of planes $L_i = 0$ will be known as the *Sylvester pentahedron* of the cubic surface.

1854: Ludwig Schläfli finds about the 27 lines from correspondence with Jacob Steiner [Graf]. In his letters he communicates to Steiner some of his results on cubic surfaces which were published later in 1858 [Schl1]. For example he shows that a general cubic surface has 36 double-sixes of lines. A *double-six* is a pair of sets of 6 skew lines (*sixes*) such that each line from one set is skew to a unique line from the other set making a bijection between the two sets. He introduces a new notation for the 27 lines $(a_i, b_i, c_{ij}, 1 \leq i < j \leq 6)$. In these notations the double-sixes are

$$\begin{pmatrix} a_1 & \dots & a_6 \\ b_1 & \dots & b_6 \end{pmatrix}, \begin{pmatrix} a_i & b_i & c_{kl} & c_{km} & c_{kn} & c_{kp} \\ a_k & b_k & c_{il} & c_{im} & c_{in} & c_{ip} \end{pmatrix}, \begin{pmatrix} a_i & a_k & a_l & c_{mn} & c_{mp} & c_{np} \\ c_{kl} & c_{ik} & c_{il} & b_p & b_n & b_m \end{pmatrix}$$

1855: Hermann Grassmann proves that three collinear nets of planes generate a cubic surface. More precisely, let $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_2$ be three general two-dimensional linear systems of planes in $\mathbb{P}^3$. Choose an isomorphism $\phi_i : \mathbb{P}^2 \to \mathcal{N}_i$ for each $i = 1, 2, 3$ and consider rational map $\Phi : \mathbb{P}^2 - \to \mathbb{P}^3$ defined by

$$\Phi(x) = \phi_1(x) \cap \phi_1(x) \cap \phi_1(x).$$

Grassmann proves that its image is a cubic surface in $\mathbb{P}^3$.

1856: Jacob Steiner [Ste] introduces 120 subsets of 9 lines which form two pairs of triples of tritangent planes (a *Triederpaar*). Each triple (a *Trieder*) consists of tritangent planes intersecting along a line not on the surface. In Schläfli's notation they are

$$\begin{matrix} a_i & b_j & c_{ij} \\ b_k & c_{jk} & a_j \\ c_{ik} & a_k & b_i \end{matrix}, \qquad \begin{matrix} c_{ij} & c_{kl} & c_{mn} \\ c_{ln} & c_{im} & c_{jk} \\ c_{km} & c_{jn} & c_{il} \end{matrix}, \qquad \begin{matrix} a_i & b_j & c_{ij} \\ b_k & a_l & c_{kl} \\ c_{ik} & c_{jl} & c_{mn} \end{matrix}$$

This gives 120 essentially different representation of a general cubic surface by an equation of the form

$$L_1L_2L_3 + L_4L_5L_6 = 0,$$

where L_i's are linear forms. The existence of such a representation was already pointed out by Cayley [Cay] and Salmon [Sal1]. In the same memoir Steiner relates the Sylvester pentahedron with the linear system of polar quadrics of the cubic surface. He also studies pencils of conics on a general cubic surface. Most of his results were stated without proof.

1860 Salmon [Sal2] and Clebsch [Cle1],[Cle2] find invariants of cubic forms in 4 variables. The ring of invariants is generated by invariants of degrees $8, 16, 24, 32, 40, 100$ where the square of the invariant of degree 100 is a polynomial in the remaining invariants. In modern terms this implies that that the GIT-quotient of the projective space of cubic surfaces modulo the group of projective transformations is isomorphic to the weighted projective space $\mathbb{P}(1,2,3,4,5)$.

1862: Fridericus August [Aug] proves that a general cubic surface can be projectively generated by three pencils of planes.

1863: Ludwig Schläfli [Schl2] classifies possible types of isolated singular points on cubic surfaces and their reality.

1863: Heinrich Schröter [Schr] shows how to obtain the 27 lines on a cubic surface obtained by Grassmann's construction and shows that there are exactly 6 triples of planes which intersect along a common line. The six lines form a half of a double-six. In modern terms this implies that the cubic surfaces are isomorphic to the blow-up of six points in projective plane.

1866 Alfred Clebsch [Cle4] proves that a general cubic surface is the image of a birational map from projective plane given by the linear system of cubics through 6 points. Using this he shows that in the lines a_i in Schläfli's notation are the images of the exceptional curves, the lines b_i are the images of conics through 5 points, and the lines c_{ij} are the images of lines through two points.

1866: Luigi Cremona and Rudolf Sturm are awarded a Prize of Berlin Academie established by Steiner for their work on cubic surfaces. A large part of their work was dedicated to supplying proof of the results announced by Steiner.

1867 The work of Sturm has been published [Stu1].

1868 The work of Cremona has been published [Cre1].

2. Projective generatedness

Let X be a cubic surface with 3 skew lines l_1, l_2, l_3. Let $\mathcal{P}_i, i = 1,2,3$ be three pencils of planes through the lines. A general plane $\pi_1 \in \mathcal{P}_1$ and a general plane $\pi_2 \in \mathcal{P}_2$ intersect along a line l. The line l intersects the cubic surface at 3 points, one on l_1, one on l_2 and the third one is the intersection point of two residual conics. Let p be the third point and $\pi_3 \in \mathcal{P}_3$ be the plane through p. Then $\pi_1 \cap \pi_1 \cap \pi_3 = \{p\}$. Consider the rational map

$$\Psi : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 - \to \mathbb{P}^3, (\pi_1, \pi_2, \pi_3) \mapsto \pi_1 \cap \pi_1 \cap \pi_3.$$

Let R be the graph of the map $(\pi_1, \pi_2) \mapsto \pi_3$, where π_3 is the plane passing through $p = \pi_1 \cap \pi_2 \cap X$. Then the image of R under Ψ is X. The subvariety $R \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ defines a relation between 3 planes which was called by August [Aug] the *duplo-projective relation*.

Cremona wants to show that August's generatedness of X implies the Grassmann generatedness by three nets of planes. This proves that every general cubic can be obtained by Grassmann's construction. Here is his beautiful argument. First, using the tritangent planes through the lines l_1, l_2, l_3, he finds the lines l_{ij} on X which intersects l_i and l_j but not three of them. Then he takes a general plane π and a triangle of lines a_1, a_2, a_3 on it. He chooses a projective isomorphism a_i with $\mathcal{P}_i$ such that the vertex $p_{ij} = a_i \cap a_j$ of the triangle corresponds to the plane in $\mathcal{P}_i$ containing the line l_{ij}. This does not yet fix the projective isomorphism. So, he chooses a point $\lambda_i \in a_i$ such that the planes π_i from $\mathcal{P}_i$ intersecting a_i at λ_i are in the correspondence R. Let $x_0 = \pi_1 \cap \pi_2 \cap \pi_3$. Now the projective isomorphisms $a_i \leftrightarrow \mathcal{P}_i$ being fixed, he takes any point $x \in \pi$ and consider the line joining x with the vertex p_{ij} of the triangle. Each of three lines obtained in this way intersects the opposite side a_k at a point $\lambda_k(x)$ and hence defines a plane $\pi_k(x) \in \mathcal{P}_k$. Let x' be the intersection point of these planes. He asks ([Cre1], p. 70) *"Quel est le lieu du point x?"* and proceeds to prove that it is equal to X. For this he shows that the locus X' is a cubic surface containing 9 lines obtained from pairwise intersections of 6 planes spanned by the lines l_i and l_{ij}. Then he tries to show that the surface X and X' has also the point x_0 in common not lying on the union of the nine lines. This of course proves that $X = X'$. He uses that the line l_i correspond to the point x taken on the side a_i of the triangle and the lines l_{ij} correspond to the vertices of the triangle. But the argument for the latter is obscure. It was noticed by Sturm in his obituary of Cremona [Stu2]. He writes that he consulted on this point with Teodor Reye, who also did not understand the argument. The gap was fixed much later by Corrado Segre [Seg2]. His proof is algebraic and very simple. One can write three plane-pencils in a duplo-projectuive relation as follows

$$\lambda_1 A_1 + \lambda_2 A_2 = 0, \mu_1 B_1 + \mu_2 B_2 = 0, \gamma_1 C_1 + \gamma_2 C_2 = 0$$

$$\lambda_1 \lambda_2 \lambda_3 = \mu_1 \mu_2 \mu_3.$$

The equation of the cubic surface becomes

$$F = A_1 B_1 C_1 + A_2 B_2 C_2 = \det \begin{pmatrix} 0 & A_1 & A_2 \\ B_2 & 0 & B_1 \\ C_1 & C_2 & 0 \end{pmatrix} = 0$$

In general, if we have a 3×3-matrix $M = (L_{ij})$ with linear forms in 4 variables t_0, t_1, t_2, t_3 as its entries, the system of linear equations

$$L_1(\lambda) = \lambda_1 L_{11} + \lambda_2 L_{12} + \lambda_3 L_{13} = 0$$
$$L_2(\lambda) = \lambda_1 L_{21} + \lambda_2 L_{22} + \lambda_3 L_{23} = 0$$
$$L_2(\lambda) = \lambda_1 L_{31} + \lambda_2 L_{32} + \lambda_3 L_{33} = 0$$

can be viewed as 3 projectively equivalent nets $V(L_\lambda)$ of planes in $\mathbb{P}^3$, a common solution is expressed by the determinant of the matrix $L = (L_{ij})$. Rewrite this system of equations by collecting terms at the unknowns t_i:

$$t_0 A_{11} + t_1 A_{12} + t_2 A_{13} + t_3 A_{14} = 0$$
$$t_0 A_{21} + t_1 A_{22} + t_2 A_{23} + t_3 A_{24} = 0$$
$$t_0 A_{31} + t_1 A_{32} + t_1 A_{33} + t_3 A_{34} = 0.$$

Now the coefficients A_{ij} are linear forms in $\lambda_1, \lambda_2, \lambda_3$. For any $(\lambda_1, \lambda_2, \lambda_3)$ satisfying $F = \det L = 0$, a solution of the rewritten system of equations defines a point in $\mathbb{P}^3$. This shows that any point on the cubic surface $F = 0$ is obtained as the intersection of three planes each from its net of planes. This gives Grassmann's representation of a cubic surface. Segre explains that Cremona's geometric argument is equivalent to this algebraic trick of rewriting the system of linear equations. Segre also points out that the determinant equation $F = \det A$ exists even for singular cubic surfaces except when it acquires a special singular point, a double rational point of type E_6 in modern terminology.

Cremona derives other nice consequences of his geometric construction of projective degeneration of a cubic surface. First he considers a plane in $\mathbb{P}^3 = \mathbb{P}(V)$ as a point in the dual space $\check{\mathbb{P}}^3 = \mathbb{P}(V^*)$. Intersection of three planes defines a rational map f from $(\check{\mathbb{P}}^3)^3$ to $\mathbb{P}^3$. Now choose a projective generation of a cubic surface X by three projectively equivalent nets of planes $V(L_i\lambda)$ as above. Assigning to a point $x \in \mathbb{P}^3$ a triple of planes Π_i from each net defines a rational map ϕ from $\mathbb{P}^3$ to $(\check{\mathbb{P}}^3)^3$. Composing it with the map f, we get a rational map

$$T : \mathbb{P}^3 - \to \mathbb{P}^3.$$

It is the famous *cubo-cubic Cremona transformation* of $\mathbb{P}^3$. It is easy to see that ϕ is defined by a linear map $V \to V^* \otimes V^* \otimes V^*$ and the map f is defined by a linear map $V^* \otimes V^* \otimes V^*$ which factors through the symmetric product $S^3(V^*)$. This shows that the map T is given by a 3-dimensional linear system of cubics. Its base locus is the pre-image of the locus of linear dependent triples of planes. It is a curve C of degree 6 of genus 3. The image of a plane is a surface of degree 3. The map is not defined on the intersection of C with the plane. This is a set of 6 points. This recovers Clebsch's result that the rational map Φ is given by a linear system of cubics through 6 points in the plane.

Using this representation Cremona easily lists all 27 lines *"Voilà donc les 27 droites de la surface F_3"* [Cre1], p. 67.

This allowed him to reconstruct Steiner's Triederpaare, tritangent planes, and double-sixes. He makes an important observation that a choice of a double-six determines two 3-dimensional linear systems of curves of degree 6 is residual to each other with respect to complete intersections of X with a quadric. As we know now such a linear system defines a determinantal representation of the cubic equation.

Steiner announces, and Cremona and Sturm prove, that each Triederpaar of tritangent planes defines two more Triederpaare such that together they consist of the 27 lines. The number of such triples is equal to 40. Each triple defines 9 tritangent planes which together contain the 27 lines. The sets of 9 tritangent planes with this property were first considered by Camille Jordan [Jor]. He called such a set a *enneaedro*. In [Cre3] Cremona divides all enneaedra in two kinds. An enneaedro of the first kind is obtained as above from a triple of Triederpaare. There are 40 of them. Only those were found by Jordan. An enneaedro of the second kind i can be uniquely divided into the union of three triples of planes, each triple belongs to a unique triple of Triederpaare as above.

3. Cremona's hexaedral equations

The last paper of Cremona devoted to cubic surfaces is his Math. Ann. paper of 1878 [Cre4]. It was shown by Reye [Rey1] that the equation of a general cubic surface F_3 can be expressed in ∞^4 ways as a sum of 6 linear forms:

$$F_3 = \sum_{i=1}^{6} L_i^3 = 0$$

(a *hexaedral equation*). The linear forms l_i define a map $\mathbb{P}^3 \to \mathbb{P}^5$ with the image a linear susbspace defined by 2 linear independent relations between the linear forms:

$$\sum_{i=0}^{5} a_i x_i = 0, \quad \sum_{i=0}^{5} b_i x_i = 0.$$

The surface F_3 is mapped to the linear section of the cubic hypersurface

$$\sum_{i=0}^{5} x_i^3 = 0.$$

Cremona shows that among ∞^4 representations of F one can always choose a representation such that one of the equations of the linear space is

$$x_0 + \ldots + x_5 = 0.$$

In fact it can be done in 36 essentially different ways corresponding to a choice of a double-six on F_3. This is done as follows. A choice of a Steiner's Triededrpaar of tritangent planes gives a Cayley-Salmon equation of X of the form $F = PQR + STU$. Then one looks for constants such that after scaling the linear forms they add up to zero. Write

$$P' = pP, Q' = qQ, R' = rR, S' = sS, T' = tT, U' = uU.$$

Since $V(F)$ is not a cone, four of the linear forms are linearly independent. After reordering the linear forms, we may assume that the forms P, Q, R, S are linearly independent. Let

$$T = aP + bQ + cR + dS, \ U = a'P + b'Q + c'R + d'S.$$

The constants p, q, r, s, t, u must satisfy the following system of equations

$$\begin{aligned} p + ta + ua' &= 0 \\ q + tb + ub' &= 0 \\ r + tc + uc' &= 0 \\ s + td + ud' &= 0 \\ pqr + stu &= 0. \end{aligned}$$

The first 4 linear equations allow us to express linearly all unknowns in terms of two, say t, u. Plugging in the last equation, we get a cubic equation in t/u. Solving it, we get a solution. Now set

$$x_1 = Q' + R' - P', \quad x_2 = R' + P' - Q', \quad x_3 = P' + Q' - R',$$
$$x_4 = T' + U' - S', \quad x_5 = U' + S' - T', \quad x_6 = S' + T' - U'.$$

One checks that these six linear forms satisfy the equations

$$x_1^3 + \ldots + x_6^3 = 0, \quad x_1 + \ldots + x_6 = 0.$$

A *Cremona hexaedral equation* defines a set of 15 lines given by the equations

$$x_i + x_j = x_k + x_l = x_m + x_n = 0, \ \sum_{i=1} a_i x_i = 0.$$

One check that the complementary set is a double-six. If two surfaces given by hexahedral equations define the same double-six, then they have common 15 lines. Obviously this is impossible. Thus the number of different hexahedral equations of X is less or equal than 36. Now consider the identity

$$(x_1+\ldots+x_6)\big((x_1+x_2+x_3)^2+(x_4+x_5+x_6)^2-(x_1+x_2+x_3)(x_4+x_5+x_6)\big)$$
$$= (x_1+x_2+x_3)^3 + (x_4+x_5+x_6)^3 = x_1^3 + \ldots + x_6^3$$
$$+3(x_2+x_3)(x_1+x_3)(x_1+x_2) + 3(x_4+x_5)(x_5+x_6)(x_4+x_6).$$

It shows that Cremona hexahedral equations define a Cayley-Salmon equation

$$(x_2+x_3)(x_1+x_3)(x_1+x_2) + (x_4+x_5)(x_5+x_6)(x_4+x_6) = 0,$$

where we have to eliminate one unknown with help of the equation $\sum a_i x_i = 0$. Applying permutations of $x_1, \ldots, x_6$, we get 10 Cayley-Salmon equations of S. Each 9 lines formed by the corresponding Steiner's Triederpaar are among the 15 lines determined by the hexahedral equation. It follows from the classification of Steiner's Triederpaare that we have 10 such pairs composed of lines c_{ij}'s. Thus a choice of Cremona hexahedral equations defines exactly 10 Cayley-Salmon equations of S. Conversely, it follows from Cremona's proof from above that each Cayley-Salmon equation gives 3 Cremona hexahedral equations (unless the cubic equation has a multiple root). Since we have 120 Cayley-Salmon equations for S we get $36 = 360/10$ hexahedral equations for S. They match with 36 double-sixes.

The cubic 3-fold $\mathcal{S}_3$ in $\mathbb{P}^5$ given by equations

$$\sum_{i=0}^{5} x_i^3 = \sum_{i=0}^{5} x_i = 0$$

is the Segre cubic primal [Seg1]. It has 10 nodes, the maximum possible for a cubic 3-fold. In 1915 Arthur Coble computes invariants of 6 points on $\mathbb{P}^1$ and proves (in modern terms) that $\mathcal{S}_3$ is isomorphic to the GIT-quotient $(\mathbb{P}^1)^6//\mathrm{SL}(2)$ [Cob]. A choice of a six skew lines defines a representation of F_3 as the blow-up of 6 points $p_1, \ldots, p_6$ in $\mathbb{P}^2$. We order them. Consider the projection of these 6 points to $\mathbb{P}^1$ from a variable point $x \in \mathbb{P}^2$. We get a map from $\mathbb{P}^2$ to S_3. Coble proves that the image is a hyperplane section $\sum_{i=0}^5 a_i x_i = 0$. This hyperplane depends only on the projective equivalence class of the six points. Also if we replace the 6 points by the six points coming from 6 skew lines which form with the previous set of skew lines a double six, we get the same hyperplane section. The moduli space of nonsingular cubic surfaces together with a choice of an ordered set of skew lines is isomorphic to the open subset of the GIT-quotient $(\mathbb{P}^2)^6//\mathrm{SL}(3)$. The involution which interchanges the 6 lines with the dual 6 lines, has the quotient isomorphic to $\mathbb{P}^4$. This is the $\mathbb{P}^4$ formed by the coefficients $(a_0, \ldots, a_5)$ in Cremona hexaedral equations!

4. Desmic quartics

Fix a tritangent plane π on X which consists of 3 lines l_1, l_2, l_3. A plane π_i through l_i has a residual conic C_i. Since the reducible curve $l_1 + l_2 + l_3$ is cut out by a plane, $C_1 + C_2 + C_3$ is cut out by a quadric. Thus we have a map

$$\mathcal{P}_1 \times \mathcal{P}_2 \times \mathcal{P}_3 \to |\mathcal{O}_{\mathbb{P}^3}(2)|,$$

where $\mathcal{P}_i$ is the pencil of planes through the line l_i. Since the map does not depend on the order of lines, it factors through a linear map $\mathbb{P}^3 \to |\mathcal{O}_{\mathbb{P}^3}(2)|$. This defines a web $\mathcal{W}$ of quadrics in $\mathbb{P}^3$. Choosing the plane π_i among the 4 tritangent planes through l_i except the plane π, gives a quadric intersecting X along the union of 6 lines. By changing π, one obtains 45 sets of 48 nonsingular quadrics with this property. Each quadric belongs to 6 sets, altogether giving 360 quadrics. This beautiful result of Steiner was given without proof. The proofs were supplied by Cremona and Sturm.

They went further by proving that the Steinerian surface of the web $\mathcal{W}$ (the set of nodes of quadrics from the web) is a quartic surface K with 12 nodes. The twelve nodes lie on X and are equal to the intersection points of 12 line-pairs corresponding to the twelve tritangent planes from above. The twelve nodes of K are the vertices of 3 desmic tetrahedra in $\mathbb{P}^3$ (*desmic* means that their equations are linear dependent). In particular, the 135 intersection points of the 27 lines on X can be grouped in 45 sets of 12 points which are the vertices of 3 desmic tetrahedra. It was proven by C. Jessop [Jes] in 1916 that the *desmic quartic* K is birationally isomorphic to the Kummer surface associated to the product of two isomorphic elliptic curves.

5. Pascal's Hexagram

A well-known Pascal's Theorem from plane projective geometry asserts that the opposite sides of a hexagon inscribed in a conic C intersect at three collinear points. In [Cre3] Cremona observes that this configuration can be obtained by projecting a cubic surface with an ordinary double point O with center at 0. The 6 lines $l_1,\ldots,l_6$ passing through O are projected to six points $p_1, \ldots, p_6$ lying on a conic. Each of 15 pairs of lines l_i, l_j defines a tritangent plane with the residual line l_{ij}. The remaining 15 tritangent planes formed by lines l_{ij}, l_{kl}, l_{mn} where the index sets have no common elements. The lines l_{ij} are projected to lines ℓ_{ij} through the points p_i, p_j. Two tritangent planes not containing the point O and with no common lines define a hexagon inscribed in the conic C. The intersection line of the two tritangent planes is projected to the line joining the intersection points of opposite sides of the hexagon. In this way a nodal cubic surface X defines 60 Pascal's Hexagrams. Next Cremona observes that all of this can be extended to a nonsingulat cubic surface. A choice of a double-six defines a Cremona hexagonal form of the equation of X. The remaining 15 lines can be indexed by the sets $\{ij, kl, mn\}$. They lie in the union of 15 tritangent planes Π_{ij} which can be indexed by the sets $\{ij\}$. Two tritangent planes Π_{ij} and Π_{kl} form a *Cremona pair* if they intersect along a line not contained in X. This terminology belongs to Reye ([Rey2], p. 218) who also calls the intersection line a *Pascal line*. There are 60 Cremona pairs. Projecting from a general point on X we obtain 60 hexagons with opposite sides intersecting at 3 points lying on a line (the projection of a Pascal line). So we get 60 Pascal's hexagrams although there is no conic in which the hexagon is inscribed!

The 15 tritangent planes Π_{ij} can be realized as the faces of 6 pentahedra P_i (the *Cremona pentahedra*), its 60 edges are the Pascal lines. All subsets of the set of 45 tritangent planes with the property that no two planes in the set intersect along a line on X were found by Eugenio Bertini [Ber].

As it turned out much later all this fascinating combinatorics of the set of 27 lines has a nice group-theoretic interpretation. It was proven by Jordan [Jor] that the group of symmetries of the incidence graph of the 27 lines is realized as the Galois group of the equation defining the 27 lines whose coefficients are rational functions in the coefficients of the equation of a general cubic surface. In modern terms this group is isomorphic to the Weyl group $W(E_6)$ of root system of type E_6. The double-sixes correspond to pairs of opposite roots, so that the stabilizer of such a pair is a maximal subgroup of index 36. The stabilizer of a line is a maximal subgroup of index 27. The stabilizer of a tritangent plane is a maximal subgroup of index 45. The stabilizer of a triple of Steiner's Triederpaare containing all 27 lines is a maximal subgroup of index 40. In this way all (except one of index 40) maximal subgroups get a geometric interpretation in terms of the geometry of 27 lines on a cubic surface.

6. Real lines on a cubic surface

The last chapter of Cremona's memoir is dedicated to the questions of reality of a cubic surface and its lines. Schläfli [Schl2] and August [Aug] have already distributed nonsingular cubic surfaces into 5 different species. For example, surfaces of the first kind have all 27 lines real. Surfaces of the second kind have 15 lines are real and 6 complex conjugate pairs of lines form a double-six. Cremona and Sturm give geometric proofs of these results. Cremona's classification takes into account also the reality of tritangent planes and double-sixes.

(1) All is real (lines, tritangent planes, double-sixes);
(2) 15 lines and 15 tritangent planes are real. There are 15 double-sixes which consist of real sixes, each six consists of 6 real lines and a pair of complex conjugate lines. One double-six consists of two complex conjugate sixes.
(3) 7 lines and 5 tritangent planes are real. There are 6 double-sixes which consist of two real sixes formed by 2 real lines and two pairs of complex conjugate lines. There are also 2 double-sixes, each consists of a pair of complex conjugate sixes.
(4) 3 lines and 7 tritangent planes are real. There is a unique double-six which consists of two real sixes formed by three pairs of complex conjugate lines. There are also 3 real-double sixes, each consists of a complex conjugate pair of sixes.
(5) 3 lines and 13 tritangent planes are real. No real double-sixes.

Cremona and Sturm prove that only surfaces of the fifth kind cannot be generated by three nets of planes as in Grassmann's construction. The reason is that on such a surface there are no sets of 6 skew lines defined over reals.

The argument of Cremona is very geometric. He first observes that a cubic surface can be reconstructed from a Triederpaar of tritangent planes and a point by using the Cayley-Salmon equation. A real surface is obtained if we find a Triederpaar and a point defined over reals. A real tritangent plane is formed by either 3 real lines or one real line and a pair of complex conjugate lines. A real Triederpaar is defined by a pair of real tritangent planes intersecting along a line not contained in the surface. Cremona proves that it is always possible to find a real Triederpaar on a real surface. Next he proceeds to consider different cases.

Case 1: The nine lines defined by a real Triederpaar are all real.

A Triederpaar contains 3 triples of skew lines. A triple of skew lines defines a unique quadric Q containing them in its ruling. The quadric Q intersect the surface along 3 lines in another ruling. If Q can be found such that the new 3 lines are all real, we get type (1) from above. If Q defines 3 lines such that one of them is real and 2 are complex conjugate, we obtain case (2).

Case 2: One of the Trieders in the pair consists of real planes, the other one consists of a real plane and two complex conujgates. This case leads to types (4) and (5).

Case 3: Each Trieder consists of one real plane and two complex conjugate planes. This leads to cases (3) and (4).

References

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